

SUPPORT VARIETIES AND REPRESENTATIONS OF TAME BASIC CLASSICAL LIE SUPERALGEBRAS

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ABSTRACT. Let κ be an algebraically closed field of characteristic $p > 3$ and \mathfrak{g} a restricted Lie superalgebra over κ . We introduce the definition of restricted cohomology for \mathfrak{g} and show its cohomology ring is finitely generated provided \mathfrak{g} is a basic classical Lie superalgebra. As a consequence, we show that the restricted enveloping algebra of a basic classical Lie superalgebra \mathfrak{g} is always wild except $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{osp}(1|2)$ or $\mathfrak{g} = \mathbf{C}(2)$. All finite dimensional indecomposable restricted representations of $\mathbf{u}(\mathfrak{osp}(1|2))$, the restricted enveloping algebra of Lie superalgebra $\mathfrak{osp}(1|2)$, are determined.

Keywords Cohomology, Support variety, Representation type, Basic classical Lie superalgebra

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1. INTRODUCTION

1.1. As generalizations and deep continuations of classical Lie theory, Lie superalgebras, supergroups and their representation theory over the field of complex numbers \mathbb{C} have been studied extensively since the classification of finite dimensional complex simple Lie superalgebras by Kac [18]. More on supergroups, supergeometry and supersymmetric theory can be found in [11, 22]. In recent years, there has been increasing interest in modular representation theory of algebraic supergroups. Especially, the modular representations of $GL(m|n)$, $Q(n)$ and ortho-symplectic supergroups have been initiated by Brundan, Kleshchev, Kujawa [5, 6, 7, 8, 20], and Shu-Wang [29]. A systemic research of modular Lie superalgebras has been started [33, 34]. In [33], the super version of the celebrated Kac-Weisfeiler Property is shown to hold for the basic classical Lie superalgebras, which by definition admit an even nondegenerate supersymmetric bilinear form and whose even subalgebras are reductive. Actually, the modular representation theory of supergroups and Lie superalgebras not only is of intrinsic interest in its own right, but also has found remarkable applications to classical mathematics. See [29] for some historical remarks.

Support varieties were introduced in the pioneering work of Alperin [1] and Carlson [9, 10] nearly 30 years ago as a method to study complexes and

resolutions of modules over group algebras. They open an algebro-geometric gate to linear representations of finite groups. Since then such ideas have been extended to restricted Lie algebras [16], Steenrod algebra [26], infinitesimal group schemes [32], arbitrary finite dimensional cocommutative Hopf algebras [17] and even to finite dimensional algebras [30]. See [31] for a nice survey on the theory of support varieties.

1.2. Up to now, we are lack of this algebro-geometric tool for modular Lie superalgebras, perhaps due to the representation theory of simple Lie superalgebras over \mathbb{C} is already very difficult and remains to be better understood. Recently, such tools were introduced for Lie superalgebras over \mathbb{C} in [4] by using so-called relative cohomology. It seems that the methods used in [4] can not be applied to positive characteristic case. The main aim of this paper is to establish a kind of definition for a support variety, which is suitable for our purpose, and give an application. At first, we realize that for any restricted Lie superalgebra \mathfrak{g} one can relate it with an ordinary Hopf algebra $\mathbf{u}(\mathfrak{g}) \rtimes \kappa\mathbb{Z}_2$ possessing equivalent representation theory as $\mathbf{u}(\mathfrak{g})$. So we can pass from “super world” to the “usual world” without losing information. Using this ordinary Hopf algebra, we can define its cohomology algebra naturally.

It is known that support varieties can be defined once the finite generation of cohomology is established, which is hard to prove in general. In this paper, we prove this finite generation property for the class of basic classical Lie superalgebras. It consists of several infinite series and 3 exceptional ones. We divide our proof in two different case $\mathfrak{g} \neq \mathbf{A}(1, 1)$ or $\mathfrak{g} = \mathbf{A}(1, 1)$. In the first case, we give a two-step filtration to reduce $\mathbf{u}(\mathfrak{g})$ to a familiar algebra whose cohomology ring is known and each of filtration involves a convergent spectral sequence. We find some permanent cycles in such spectral sequences and apply a lemma cited from [23] to conclude finite generation. To give the filtration, a new kind of PBW basis are developed. We put the case $\mathfrak{g} = \mathbf{A}(1, 1)$ in a bigger context, in which all $\mathbf{u}(\mathfrak{g})$ are equipped with a nice filtration similar to the coradical filtration of a coalgebra. Through this one-step filtration, we can reduce $\mathbf{u}(\mathbf{A}(1, 1))$ to a familiar algebra already. Then the same idea developed in the first case can be applied.

One central question in the modern representation theory of algebras is the determination of the representation type. By Drozd’s fundamental trichotomy [12], finite dimensional algebras over an algebraically closed field may be subdivided into the disjoint classes of representation finite, tame and wild algebras. As an application of support varieties we built, we will prove all $\mathbf{u}(\mathfrak{g})$ are wild with only three exceptions: $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{osp}(1|2), \mathbf{C}(2)$. The case $\mathbf{C}(2)$ is conjectured to be wild and we have known $\mathbf{u}(\mathfrak{sl}_2)$ and

$\mathbf{u}(\mathfrak{osp}(1|2))$ are tame. Inspired by the similarity between \mathfrak{sl}_2 and $\mathfrak{osp}(1|2)$ and for further understanding of the representations of modular Lie superalgebras, all finite dimensional restricted indecomposable $\mathfrak{osp}(1|2)$ -modules are also characterized.

The paper is organized as follows. All subsidiary results to prove the finite generation of cohomology algebras are builded in Section 2. Especially, a new kind of PBW basis suitable for our purpose and some filtrations are given. Section 3 is to give the proof of finite generation. The definition of a support variety is given in Section 4. Moreover, its connections with complexity and representation type are established. As the final conclusion of this section, the representation type of any $\mathbf{u}(\mathfrak{g})$ is determined except the case $\mathbf{C}(2)$, which is conjectured to be a wild algebra. In the last section of this paper, a complete list of all finite dimensional restricted indecomposable $\mathfrak{osp}(1|2)$ -modules up to isomorphism is formulated.

2. PRELIMINARIES

Throughout of this paper, κ is an algebraically closed field of characteristic $p \neq 0$ and $p > 3$ is always assumed unless stated otherwise. All spaces are κ -spaces. All modules are left modules.

2.1. Hopf algebras in Yetter-Drinfeld categories. Let J be a Hopf algebra with bijective antipode and ${}^J\mathcal{YD}$ the category of the Yetter-Drinfeld modules with left J -module action and left J -comodule coaction. It is naturally forms a braided monoidal category with the braiding

$$c_{M,N} : M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto \sum n_0 \otimes n_{-1} \cdot m,$$

where $n \mapsto \sum n_{-1} \otimes n_0$, $N \rightarrow J \otimes N$ denotes the comodule structure, as usual. Let A be a *braided Hopf algebra* in ${}^J\mathcal{YD}$. By definition, it is an algebra as well as coalgebra in ${}^J\mathcal{YD}$ such that its comultiplication and counit are algebra morphism, and such that the identity morphism has a convolution inverse in ${}^J\mathcal{YD}$. When we say that the comultiplication $\Delta : A \rightarrow A \otimes A$ should be an algebra morphism, the braiding defined as above arises in the definition of the algebra structure of $A \otimes A$ and so A is not an ordinary Hopf algebra in general. Through the Radford-Majid bosonization [21, 27], it gives rise to an ordinary Hopf algebra $A \rtimes J$. As an algebra, this is the smash product $A \# J$, and it is the smash coproduct as a coalgebra.

Lemma 2.1. *Let J be a Hopf algebra with bijective antipode and A a braided Hopf algebra in ${}^J\mathcal{YD}$. Then the cohomology ring $H^*(A, \kappa) := \bigoplus_{i \geq 0} \text{Ext}_A^i(\kappa, \kappa)$ is a braided graded commutative algebra in ${}^J\mathcal{YD}$.*

Proof. By Theorem 3.12 in [23], the Hochschild cohomology ring

$$\mathrm{HH}^*(A, \kappa) := \bigoplus_{i \geq 0} \mathrm{Ext}_{A \otimes A^{op}}^i(A, \kappa)$$

is a braided graded commutative algebra in ${}^J\mathcal{YD}$. By the standard bar resolution for computing these extension groups, one can see that $\mathrm{Ext}_A^i(\kappa, \kappa) \cong \mathrm{Ext}_{A \otimes A^{op}}^i(A, \kappa)$ for $i \geq 0$ (see also subsection 2.4 in [23]). The proof is complete. \square

2.2. Cohomology of restricted Lie superalgebras. We fix some notions at first. By definition, a superalgebra is nothing but a \mathbb{Z}_2 -graded algebra. By forgetting the grading we may consider any superalgebra A as a usual algebra and this algebra will be denoted by $|A|$. For any two \mathbb{Z}_2 -graded vector spaces V, W , we use $\mathrm{Hom}_\kappa(V, W)$ to represent the set of all linear maps from V to W and $\underline{\mathrm{Hom}}_\kappa(V, W)$ to denote that of all even linear maps.

Now let $A = A_0 \oplus A_1$ be a superalgebra. Then there is a natural action of $\mathbb{Z}_2 = \langle g | g^2 = 1 \rangle$ on A given by

$$g \cdot a = a, \quad g \cdot b = -b, \quad \text{for } a \in A_0, b \in A_1.$$

Note that this definition makes sense as stated only for homogeneous elements, it should be interpreted via linearity in the general case. Thus A is a $\kappa\mathbb{Z}_2$ -module algebra (for definition, see Section 4.1 in [25]) and the smash product $A \# \kappa\mathbb{Z}_2$ is a usual algebra. We use $A\text{-smod}$ to denote the category of all finitely generated left A -supermodules with even homomorphisms and $A \# \kappa\mathbb{Z}_2\text{-mod}$ the usual finitely generated left $A \# \kappa\mathbb{Z}_2$ -modules category.

Lemma 2.2. *Let A be a superalgebra. Then $A\text{-smod}$ is equivalent to $A \# \kappa\mathbb{Z}_2\text{-mod}$.*

Proof. Let $M = M_0 \oplus M_1$ be an A -supermodule and $g \in \mathbb{Z}_2$ the generator of \mathbb{Z}_2 . Through assigning $g \cdot m_0 := m_0$, $g \cdot m_1 := -m_1$ for $m_0 \in M_0$, $m_1 \in M_1$, M is a $\kappa\mathbb{Z}_2$ -module. Now just define the action of $A \# \kappa\mathbb{Z}_2$ on M through $(a \otimes g) \cdot m := a \cdot (g \cdot m)$ for $a \in A$ and $m \in M$. To show it is indeed an $A \# \kappa\mathbb{Z}_2$ -module, one need verify the equality

$$(1 \otimes g)(a \otimes 1) \cdot m = ((g \cdot a) \otimes g) \cdot m, \quad (*)$$

for $a \in A$ and $m \in M$. It is not hard to see that this is equivalent to the fact $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$.

Conversely, let M be an $A \# \kappa\mathbb{Z}_2$ -module. Since the characteristic of κ is not equal to 2, $\kappa\mathbb{Z}_2$ is semisimple. Therefore, $M = M_0 \oplus M_1$ with $M_0 = \{m \in M | g \cdot m = m\}$ and $M_1 = \{m \in M | g \cdot m = -m\}$. Also, the equality $(*)$ implies that $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$. Thus M is an A -supermodule.

At last, it is clear that $\underline{\mathrm{Hom}}_A(-, -) = \mathrm{Hom}_{A \# \kappa\mathbb{Z}_2}(-, -)$. The lemma is proved. \square

Now we specialize this simple observation to the case of restricted enveloping algebras of restricted Lie superalgebras.

Definition 2.3. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called a *restricted Lie superalgebra*, if there is a p th map $\mathfrak{g}_0 \rightarrow \mathfrak{g}_0$, denoted as $^{[p]}$, satisfying

- (a) $(cx)^{[p]} = c^p x^{[p]}$ for all $c \in k$ and $x \in \mathfrak{g}_0$,
- (b) $[x^{[p]}, y] = (adx)^p(y)$ for all $x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}$,
- (c) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ for all $x, y \in \mathfrak{g}_0$ where s_i is the coefficient of λ^{i-1} in $(ad(\lambda x + y))^{p-1}(x)$.

In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module by the adjoint action of the even subalgebra. All the Lie (super)algebras in this paper will be assumed to be restricted. For a restricted Lie superalgebra \mathfrak{g} , $U(\mathfrak{g})$ is denoted to be its universal enveloping algebra and $\mathbf{u}(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} | x \in \mathfrak{g}_0)$ its restricted enveloping algebra. The following is a consequence of PBW theorem for $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$.

Lemma 2.4. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and x_1, \dots, x_s a basis of \mathfrak{g}_1 , y_1, \dots, y_t a basis of \mathfrak{g}_0 . Then

- (1) $U(\mathfrak{g})$ has a basis

$$\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} | b_i \in \mathbb{N}, a_j = 0, 1 \text{ for all } i, j\}.$$

- (2) $\mathbf{u}(\mathfrak{g})$ has a basis

$$\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} | 0 \leq b_i < p, a_j = 0, 1 \text{ for all } i, j\}.$$

The following proposition gives new kinds of PBW basis, which are suitable for our purpose.

Proposition 2.5. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and x_1, \dots, x_s a basis of \mathfrak{g}_1 in which we assume $[x_i, x_i] = 0$ for $i \leq s_1$ and $z_j := [x_j, x_j] \neq 0$ for $s_1 < j \leq s$. Assume that z_{s_1+1}, \dots, z_s are linear independent and denote the subspace of \mathfrak{g}_0 spanned by them by V . Let W be a subspace of \mathfrak{g}_0 such that $\mathfrak{g}_0 = W \oplus V$ and y_1, \dots, y_{t_1} be a basis of W . Then

- (1) $U(\mathfrak{g})$ has a basis consisting of

$$x_1^{a_1} \cdots x_{s_1}^{a_{s_1}} x_{s_1+1}^{b_1} \cdots x_s^{b_{s-s_1}} y_1^{c_1} \cdots y_{t_1}^{c_{t_1}}$$

where $0 \leq a_i < 2$, $b_j, c_k \in \mathbb{N}$ for all i, j, k .

- (2) $\mathbf{u}(\mathfrak{g})$ has a basis consisting of

$$x_1^{a_1} \cdots x_{s_1}^{a_{s_1}} x_{s_1+1}^{b_1} \cdots x_s^{b_{s-s_1}} y_1^{c_1} \cdots y_{t_1}^{c_{t_1}}$$

where $0 \leq a_i < 2$, $0 \leq b_j < 2p$, $0 \leq c_k < p$ for all i, j, k .

Proof. We only prove (2) since (1) can be proved similarly. By assumption the set $\{z_i, y_j | s_1 < i \leq s, 0 \leq j \leq t_1\}$ is a basis of \mathfrak{g}_0 . Owing to Lemma 2.4 (2),

$$\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_{t_1+s-s_1}^{b_{t_1+s-s_1}} | 0 \leq b_i < p, a_j = 0, 1 \text{ for all } i, j\}$$

is a basis of $\mathbf{u}(\mathfrak{g})$ where we set $y_{t_1+i} := z_i$ ($s_1 + 1 \leq i \leq s$) for consistence. By the proof of the PBW theorem, there is no any restriction on the order of elements we choose and thus the following elements also form a basis of $\mathbf{u}(\mathfrak{g})$:

$$(2.1) \quad x_1^{a_1} \cdots x_{s_1}^{a_{s_1}} x_{s_1+1}^{a_{s_1+1}} z_{s_1+1}^{b_{s_1+1}} \cdots x_s^{a_s} z_s^{b_s} y_1^{b_1} \cdots y_{t_1}^{b_{t_1}}$$

where $0 \leq b_i < p$, $a_j = 0, 1$ for all i, j . Since

$$z_i = [x_i, x_i] = 2x_i^2$$

in $\mathbf{u}(\mathfrak{g})$ for $s_1 + 1 \leq i \leq s$, the set $\{x_i^{a_i} z_i^{b_i} | a_i = 0, 1, 0 \leq b_i < p\} = \{a(m_i)x_i^{m_i} | 0 \leq m_i < 2p, \text{ some } 0 \neq a(m_i) \in \kappa\}$. So we can abbreviate elements of (2.1) and get the ones described in the proposition. The conclusion is proved. \square

Both $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ are super cocommutative Hopf algebras. Thus they are braided Hopf algebras in ${}^{\kappa\mathbb{Z}_2}_{\kappa\mathbb{Z}_2}\mathcal{YD}$. In particular, $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ is an ordinary algebra. Actually, it is a Hopf algebra by above subsection. Let M, N be two $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ -modules and $P_\bullet \rightarrow M$ be a projective resolution of M . Define

$$H_{\mathbf{u}(\mathfrak{g})}^i(M, N) := \text{Ext}_{\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2}^i(M, N) = H^i(\text{Hom}_{\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2}(P_\bullet, N)),$$

$$H^i(\mathbf{u}(\mathfrak{g}), M) := \text{Ext}_{\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2}^i(\kappa, M) \text{ and}$$

$$H^i(\mathbf{u}(\mathfrak{g}), \kappa) := \text{Ext}_{\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2}^i(\kappa, \kappa)$$

for $i \geq 0$, where κ is the trivial $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ -module with the action gotten through the counit $\varepsilon : \mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2 \rightarrow \kappa$.

Remark 2.6. By Lemma 2.2, this is equivalent to say that we consider the restricted cohomology of a restricted Lie superalgebra \mathfrak{g} exactly in the category $\mathbf{u}(\mathfrak{g})$ -smod. That is, we only consider even homomorphisms. This is totally different with the relative cohomology defined in [4], where the authors indeed bring all homomorphisms into consideration.

For any coalgebra C , we denote $\text{Ker } \varepsilon$ by C^+ as usual. Also, as a usual algebra $|\mathbf{u}(\mathfrak{g})|$ has its usual cohomology $H^i(|\mathbf{u}(\mathfrak{g})|, N)$ for any $|\mathbf{u}(\mathfrak{g})|$ -module N . For any Hopf algebra H and H -module M , we define $M^H := \{m \in M | h \cdot m = \varepsilon(h)m, \text{ for all } h \in H\}$.

Lemma 2.7. *Let N be a $\mathbf{u}(\mathfrak{g})$ -supermodule. Then for any natural number i ,*

$$H^i(\mathbf{u}(\mathfrak{g}), N) \cong H^i(|\mathbf{u}(\mathfrak{g})|, N)^{\kappa\mathbb{Z}_2}.$$

Proof. At first, we prove the conclusion in the case $N = \kappa$. Note that $|\mathbf{u}(\mathfrak{g})|^+$ is the augmentation ideal of $|\mathbf{u}(\mathfrak{g})|$. Now consider the bar resolution of κ

$$(2.2) \quad \cdots \rightarrow |\mathbf{u}(\mathfrak{g})| \otimes (|\mathbf{u}(\mathfrak{g})|^+)^{\otimes 2} \xrightarrow{d_2} |\mathbf{u}(\mathfrak{g})| \otimes |\mathbf{u}(\mathfrak{g})|^+ \xrightarrow{d_1} |\mathbf{u}(\mathfrak{g})| \xrightarrow{\varepsilon} \kappa \rightarrow 0,$$

where $d_i(a_0 \otimes \cdots \otimes a_i) = \sum_{j=0}^{i-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i$. Thus every differential map d_i is indeed an even homomorphism. Applying $\text{Hom}_{|\mathbf{u}(\mathfrak{g})|}(-, \kappa)$, one get

$$(2.3) \quad 0 \rightarrow \text{Hom}_{\kappa}(\kappa, \kappa) \xrightarrow{\delta_0} \text{Hom}_{\kappa}(|\mathbf{u}(\mathfrak{g})|^+, \kappa) \xrightarrow{\delta_1} \text{Hom}_{\kappa}(|\mathbf{u}(\mathfrak{g})|^+{}^{\otimes 2}, \kappa) \xrightarrow{\delta_2} \cdots,$$

where $\delta_i = d_i^*$. By definition, $H^i(|\mathbf{u}(\mathfrak{g})|, \kappa) = \text{Ker } \delta_i / \text{Im } \delta_{i-1}$. Meanwhile, $H^i(\mathbf{u}(\mathfrak{g}), \kappa)$ is exactly the i th cohomology of the following complex

$$0 \rightarrow \underline{\text{Hom}}_{\kappa}(\kappa, \kappa) \xrightarrow{\delta_0} \underline{\text{Hom}}_{\kappa}(\mathbf{u}(\mathfrak{g})^+, \kappa) \xrightarrow{\delta_1} \underline{\text{Hom}}_{\kappa}((\mathbf{u}(\mathfrak{g})^+)^{\otimes 2}, \kappa) \xrightarrow{\delta_2} \cdots.$$

Since $\text{Hom}_{\kappa}((\mathbf{u}(\mathfrak{g})^+)^{\otimes i}, \kappa)^{\kappa\mathbb{Z}_2} = \text{Hom}_{\kappa\mathbb{Z}_2}((\mathbf{u}(\mathfrak{g})^+)^{\otimes i}, \kappa) = \underline{\text{Hom}}_{\kappa}((\mathbf{u}(\mathfrak{g})^+)^{\otimes i}, \kappa)$, $H^i(\mathbf{u}(\mathfrak{g}), \kappa) \cong H^i(|\mathbf{u}(\mathfrak{g})|, \kappa)^{\kappa\mathbb{Z}_2}$.

In general, for any $\mathbf{u}(\mathfrak{g})$ -supermodule N , one can apply $\text{Hom}_{|\mathbf{u}(\mathfrak{g})|}(-, N)$ to (2.2) to get a similar complex like (2.3). Using totally the same argument as κ , one can get the desired conclusion. \square

The following result is a direct consequence of Lemma 2.1 by noting that $\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2$ is an ordinary Hopf algebra.

Corollary 2.8. *Let M be an $\mathbf{u}(\mathfrak{g})$ -supermodule. Then under cup product, $H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa) := \bigoplus_{i \geq 0} H^{2i}(\mathbf{u}(\mathfrak{g}), \kappa)$ is a commutative algebra and $H^*(\mathbf{u}(\mathfrak{g}), M) := \bigoplus_{i \geq 0} H^i(\mathbf{u}(\mathfrak{g}), M)$ is an $H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa)$ -module.*

2.3. Basic classical Lie superalgebras.

Definition 2.9. A Lie superalgebra is a *basic classical Lie superalgebra* if it admits an even nondegenerate supersymmetric bilinear form and its even subalgebra is reductive.

In the following, we only deal with basic classical Lie superalgebras unless we state otherwise. We recall the list of basic classical Lie superalgebra (see [18, 33]). They are four infinite series $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$ and three exceptional versions $\mathbf{D}(2, 1; \alpha)$, $\mathbf{G}(3)$, $\mathbf{F}(4)$ for $\alpha \in \kappa \setminus \{0, 1\}$. They are still simple Lie superalgebras even the characteristic of base field is not zero. One merit of a basic classical Lie superalgebra \mathfrak{g} is that it admits nice root space decompositions:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

such that

- (i) \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} ;
- (ii) $\dim_{\kappa} \mathfrak{g}_{\alpha} = 1$ for $\alpha \in \Phi$ except for $\mathbf{A}(1, 1)$;

(iii) Except for $\mathbf{A}(1, 1)$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if and only if $\alpha, \beta, \alpha + \beta \in \Phi$.

See Section 2.5.3 in [18] for details by noting we still can do such decompositions in positive characteristic case. In order to discriminate different root in characteristic p case, we always assume $p > 3$. Also, we fix a root decomposition just as described in Section 2.5.4 in [18] from now on. Φ is called a *root supersystem* of \mathfrak{g} . Clearly, $\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 is the root system of \mathfrak{g}_0 and Φ_1 is the system of weights of the representation of \mathfrak{g}_0 on \mathfrak{g}_1 . Φ_0 is called the *even system* and Φ_1 the *odd system*. Define

$$\Phi_{11} := \{\alpha \in \Phi_1 \mid [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0\}, \quad \Phi_{12} := \{\alpha \in \Phi_1 \mid [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \neq 0\}.$$

By observing the root supersystem of $\mathbf{B}(m, n)$, $\Phi_{12} \neq \emptyset$ in general.

Lemma 2.10. *Let \mathfrak{g} be a basic classical Lie superalgebra. Then for any $\alpha \in \Phi_0$ and $x \in \mathfrak{g}_\alpha$, $x^p = 0$ in $\mathbf{u}(\mathfrak{g})$.*

Proof. This should be known, but the author can not find suitable reference. So we give a short proof here. It is known that the even part \mathfrak{g}_0 of a basic classical Lie superalgebra \mathfrak{g} is a direct sum of some Lie algebras of types $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n, \mathbf{G}_2$ and κ . Therefore there is no harm to assume that \mathfrak{g}_0 is a simple Lie algebra of type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ or \mathbf{G}_2 . So \mathfrak{g}_0 is generated by \mathfrak{sl}_2 -triples $\{e_i, f_i, h_i \mid i \in I\}$. Thus firstly we assume that $x = e_i$ or $x = f_i$ for some i . Say, $x = e_i$. Note that e_i commutes with all f_j unless $j = i$ and in this case $\text{ad}(e_i)^3(f_i) = 0$. So $\text{ad}(e_i)^p(f_j) = 0$ for all $j \in I$. From Serre's relation, $\text{ad}(e_i)^{1-a_{ij}}(e_j) = 0$ where $(a_{ij})_{I \times I}$ is the Cartan matrix of \mathfrak{g}_0 . This implies $\text{ad}(e_i)^p(e_j) = 0$ for all $j \in I$ since $p > 1 - a_{ij}$ by our assumption on p . Also clearly $\text{ad}(e_i)^p(h_j) = 0$ for all $j \in I$. By the definition of restricted Lie algebra, $x^{[p]}$ lies in the center of \mathfrak{g}_0 and so $x^{[p]} = 0$, which implies $x^p = 0$ in $\mathbf{u}(\mathfrak{g})$ too. The case $x = f_i$ can be proved similarly. For general $x \in \mathfrak{g}_\alpha$, it is well known that up to a scalar we can get x by applying the Lie algebra automorphisms $\tau_j := \exp(\text{ad}(e_j))\exp(\text{ad}(-f_j))\exp(\text{ad}(e_j))$ iteratively to some e_i or f_i . Thus $x^{[p]}$ lies in the center too. \square

There is a filtration on $\mathbf{u}(\mathfrak{g})$ with degrees

$$\deg_1(\mathfrak{h}) = 0, \quad \deg_1\left(\bigoplus_{\alpha \in \Phi_1} \mathfrak{g}_\alpha\right) = 1, \quad \deg_1\left(\bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha\right) = 2.$$

The associated graded algebra is denoted by $\text{Gr}^1(\mathbf{u}(\mathfrak{g}))$. It is still a super cocommutative Hopf algebra. It is not hard to see that there is a natural projection from $\text{Gr}^1(\mathbf{u}(\mathfrak{g}))$ to $\mathbf{u}(\mathfrak{h})$ and thus there is a subsuperalgebra $R_{\mathfrak{g}}$ such that

$$\text{Gr}^1(\mathbf{u}(\mathfrak{g})) = R_{\mathfrak{g}} \# \mathbf{u}(\mathfrak{h}).$$

Actually, $R_{\mathfrak{g}}$ is the graded subalgebra generated by $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

For any set S , its cardinal number is denoted by $S^\#$. Assume that $\mathfrak{g} \neq \mathbf{A}(1, 1)$. Then by property (ii) of the root space decomposition, up to scalars there is a unique nonzero element x_α belonging to \mathfrak{g}_α .

Lemma 2.11. *Assume that $\mathfrak{g} \neq \mathbf{A}(1, 1)$ and let x_α defined as above. Then the graded algebra $R_{\mathfrak{g}}$ has the following PBW basis consisting of elements*

$$(2.4) \quad x_{\alpha_1}^{a_1} \cdots x_{\alpha_r}^{a_r} x_{\beta_1}^{b_1} \cdots x_{\beta_s}^{b_s} x_{\gamma_1}^{c_1} \cdots x_{\gamma_t}^{c_t}$$

where $\alpha_i \in \Phi_{11}, \beta_j \in \Phi_{12}, \gamma_k \in \Phi_0$, $r = \Phi_{11}^\#, s = \Phi_{12}^\#, t = \Phi_0^\# - \Phi_{12}^\#$ and $0 \leq a_i < 2, 0 \leq b_j < 2p, 0 \leq c_k < p$ for $1 \leq i \leq r, 1 \leq j \leq s, 1 \leq k \leq t$.

Proof. Under the grading Gr^1 , one can see that

$$[\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha] \subseteq \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha.$$

So to show the conclusion, we can assume that $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ is a Lie subsuperalgebra of \mathfrak{g} . Being living in different root spaces, $\{[x_{\beta_j}, x_{\beta_j}] | 1 \leq j \leq s\}$ are linear independent. So Proposition 2.5 can be applied and thus the set of elements in (2.4) forms a basis of $\mathfrak{u}(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$. Clearly such elements are homogeneous in $R_{\mathfrak{g}}$ and so they also give a basis of $R_{\mathfrak{g}}$. \square

Throughout the following of this subsection, we always assume that $\mathfrak{g} \neq \mathbf{A}(1, 1)$. In order to reduce $R_{\mathfrak{g}}$ to a familiar algebra, we introduce another kind of filtration on $R_{\mathfrak{g}}$. To attack it, the degree of an element in (2.4) is defined to be

$$\deg_2(x_{\alpha_1}^{a_1} \cdots x_{\gamma_s}^{c_t}) = (a_1, \dots, a_r, b_1, \dots, b_s, c_1, \dots, c_t) \in \mathbb{N}^{\Phi^\#}$$

and totally order the elements (2.4) lexicographically by setting

$$(1, 0, \dots, 0) > \cdots > (0, 1, \dots, 0) > \cdots > (0, 0, \dots, 1).$$

For convenience and consistence, we set $\alpha_{r+i} := \beta_i$ ($1 \leq i \leq s$) and $\alpha_{r+s+i} := \gamma_i$ ($1 \leq i \leq t$).

Lemma 2.12. *Under the total order defined above, for all $i < j$,*

$$\deg_2([x_{\alpha_i}, x_{\alpha_j}]) < \deg_2(x_{\alpha_i} x_{\alpha_j})$$

unless $[x_{\alpha_i}, x_{\alpha_j}] = 0$.

Proof. It is not hard to see that any $x \in \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha$ actually lies in the center of $R_{\mathfrak{g}}$. So to show the lemma, one can assume that both x_{α_i} and x_{α_j} are odd elements and $[x_{\alpha_i}, x_{\alpha_j}] \neq 0$. Now $[x_{\alpha_i}, x_{\alpha_j}]$ lies in \mathfrak{g}_0 automatically and thus either $[x_{\alpha_i}, x_{\alpha_j}] = cx_{\alpha_l}$ for $l > j$ and $0 \neq c \in \kappa$ or $[x_{\alpha_i}, x_{\alpha_j}] = d[x_{\alpha_k}, x_{\alpha_k}]$ for some odd element with $[x_{\alpha_k}, x_{\alpha_k}] \neq 0$ and $0 \neq d \in \kappa$. In the first case, the conclusion is clear. In the second case, we still need to consider two cases: $[x_{\alpha_i}, x_{\alpha_i}] = [x_{\alpha_j}, x_{\alpha_j}] = 0$ or either of them is not zero. Also, the first case

implies that $j < k$ by the PBW basis we choose and thus the conclusion is proved. By property (iii) of the root space decomposition, $\alpha_i + \alpha_j$ is still a root and it is equals to $2\alpha_k$ by assumption. Comparing with the root supersystem listed in Section 2.5.4 in [18], this is happened only in the case $[x_{\alpha_i}, x_{\alpha_i}] = [x_{\alpha_j}, x_{\alpha_j}] = 0$. \square

By Lemma 2.12, the above ordering induces a filtration on $R_{\mathfrak{g}}$. The associated graded algebra is denoted by $\text{Gr}^2(R_{\mathfrak{g}})$. It is generated by $\{x_{\alpha_i} | 1 \leq i \leq \Phi^\# - \Phi_{12}^\#\}$ with relations

$$(2.5) \quad [x_{\alpha_i}, x_{\alpha_j}] = 0 \text{ for } i \neq j, \quad x_{\alpha_i}^{N_i} = 0$$

where

$$N_i = \begin{cases} 2, & 0 \leq i \leq \Phi_{11}^\# \\ 2p, & \Phi_{11}^\# + 1 \leq i \leq \Phi_{11}^\# + \Phi_{12}^\# \\ p, & \Phi_{11}^\# + \Phi_{12}^\# + 1 \leq i \leq \Phi^\# - \Phi_{12}^\#. \end{cases}$$

Note that $\text{Gr}^2(R_{\mathfrak{g}})$ inherits the action of $\mathbf{u}(\mathfrak{h})$ from that on $R_{\mathfrak{g}}$ naturally, define

$$\text{Gr}^2(\mathbf{u}(\mathfrak{g})) := \text{Gr}^2(R_{\mathfrak{g}}) \# \mathbf{u}(\mathfrak{h}).$$

2.4. Spectral sequences and finite generation. We will see in the next section that there are some convergent spectral sequences associated to the filtrations given in Subsection 2.3. The following lemma, which is essentially used in this paper, is given in [23] as its Lemma 2.5. Recall that an element $a \in E_r^{p,q}$ is called a *permanent cycle* if $d_i(a) = 0$ for all $i \geq r$.

Lemma 2.13. (1) Let $E_1^{p,q} \Rightarrow E_\infty^{p,q}$ be a multiplicative spectral sequence of κ -algebras concentrated in the half plane $p+q \geq 0$, and let $A^{*,*}$ be a bigraded commutative κ -algebra concentrated in even (total) degrees. Assume that there exists a bigraded map of algebras $\varphi : A^{*,*} \rightarrow E_1^{*,*}$ such that

- (i) φ makes $E_1^{*,*}$ into a Noetherian $A^{*,*}$ -module, and
- (ii) the image of $A^{*,*}$ in $E_1^{*,*}$ consists of permanent cycles.

Then E_∞^* is a Noetherian module over $\text{Tot}(A^{*,*})$.

(2) Let $\tilde{E}_1^{p,q} \Rightarrow \tilde{E}_\infty^{p,q}$ be a spectral sequence that is a bigraded module over the spectral sequence $E^{*,*}$. Assume that $\tilde{E}_1^{*,*}$ is a Noetherian module over $A^{*,*}$ where $A^{*,*}$ acts on $\tilde{E}_1^{*,*}$ via the map φ . Then \tilde{E}_∞^* is a finitely generated E_∞^* -module.

3. FINITE GENERATION

The following conclusion is one of main results of this paper.

Theorem 3.1. Let \mathfrak{g} be one of basic classical Lie superalgebras over κ and $\mathbf{u}(\mathfrak{g})$ its restricted enveloping algebra. Then

- (1) the algebra $H^*(\mathbf{u}(\mathfrak{g}), \kappa) := \bigoplus_{i \geq 0} H^i(\mathbf{u}(\mathfrak{g}), \kappa)$ is finitely generated.

(2) $H^*(\mathfrak{u}(\mathfrak{g}), M)$ is a finitely generated module over $H^*(\mathfrak{u}(\mathfrak{g}), \kappa)$ for M a finitely generated $\mathfrak{u}(\mathfrak{g})$ -supermodule.

We will divide the proof into two cases: $\mathfrak{g} \neq \mathbf{A}(1, 1)$ or $\mathfrak{g} = \mathbf{A}(1, 1)$. The basic idea of the proof is to modify the procedure developed in [23] into our cases by applying preliminary results gotten in Section 2. Firstly, $\mathfrak{g} \neq \mathbf{A}(1, 1)$ is assumed until Subsection 3.4.

3.1. Cohomology of $\text{Gr}^2(\mathfrak{u}(\mathfrak{g}))$. The algebraic structure of $\text{Gr}^2(R_{(\mathfrak{g})})$ has been described clearly in (2.5). Recall that we denote the usual algebra of superalgebra A by $|A|$. For continuation, we write the algebraic structure of $|\text{Gr}^2(R_{(\mathfrak{g})})|$ again as follows: it is generated by $\{x_{\alpha_i} | 1 \leq i \leq \Phi^\# - \Phi_{12}^\#\}$ with relations

$$(3.1) \quad x_{\alpha_i} x_{\alpha_j} = \begin{cases} -x_{\alpha_j} x_{\alpha_i}, & 1 \leq i < j \leq \Phi_1^\# \\ x_{\alpha_j} x_{\alpha_i}, & 1 \leq i < j \text{ and } j > \Phi_1^\#, \end{cases} \quad x_{\alpha_i}^{N_i} = 0$$

where

$$N_i = \begin{cases} 2, & 0 \leq i \leq \Phi_{11}^\# \\ 2p, & \Phi_{11}^\# + 1 \leq i \leq \Phi_{11}^\# + \Phi_{12}^\# \\ p, & \Phi_{11}^\# + \Phi_{12}^\# + 1 \leq i \leq \Phi^\# - \Phi_{12}^\#. \end{cases}$$

The algebra $|\text{Gr}^2(R_{(\mathfrak{g})})|$ is a special case of so-called *quantum complete intersection algebras*: Let N be positive integer, and for each $i \in \{1, \dots, N\}$, N_i be an integer greater than 1. Let $q_{ij} \in \kappa^* = \kappa \setminus \{0\}$ for $1 \leq i < j \leq N$. Define S to be the κ -algebra generated by x_1, \dots, x_N subject to the relations

$$(3.2) \quad x_i x_j = q_{ij} x_j x_i \text{ for all } i < j \text{ and } x_i^{N_i} = 0 \text{ for all } i.$$

S is called a quantum complete intersection algebra. For such S , its cohomology ring $H^*(S, \kappa) = \bigoplus_{i \geq 0} \text{Ext}_S^i(\kappa, \kappa)$ was determined in Section 4 of [23]. For completeness and consistence of the paper, let us sketch it.

Let K_\bullet be the following complex of free S -modules. For each N -tuple (a_1, \dots, a_N) of nonnegative integers, let $\Psi(a_1, \dots, a_N)$ be a free generator in degree $a_1 + \dots + a_N$. Then define $K_n = \bigoplus_{a_1 + \dots + a_N = n} S \Psi(a_1, \dots, a_N)$. For each $i \in \{1, \dots, N\}$, let $\sigma_i, \tau_i : \mathbb{N} \rightarrow \mathbb{N}$ be the function defined by

$$\sigma_i(a) = \begin{cases} 1, & a \text{ is odd} \\ N_i - 1, & a \text{ is even,} \end{cases}$$

and $\tau_i(a) = \sum_{j=1}^a \sigma_i(a_j)$ for $a \geq 1$, $\tau_i(0) = 0$. Let

$$d_i(\Psi(a_1, \dots, a_N)) = \left(\prod_{l < i} (-1)^{a_l} q_{li}^{\sigma_i(a_i) \tau_l(a_l)} \right) x_i^{\sigma_i(a_i)} \Psi(a_1, \dots, a_i - 1, \dots, a_N)$$

if $a_i > 0$, and $d_i(\Psi(a_1, \dots, a_N)) = 0$ if $a_i = 0$. Extend each d_i to an S -module homomorphism and set

$$d = d_1 + \dots + d_N.$$

It is shown in Section 4 of [23] that (K_\bullet, d) is a resolution of κ .

From this resolution, one can compute $\text{Ext}_S^i(\kappa, \kappa)$. Applying $\text{Hom}_S(-, \kappa)$ to K_\bullet , the induced differential d^* is the zero map (since $x_i^{\sigma_i(a_i)}$ is always in the augmentation ideal) and thus the cohomology is just the complex $\text{Hom}_S(K_\bullet, \kappa)$. Now let $\xi_i \in \text{Hom}_S(K_2, \kappa)$, $\eta_i \in \text{Hom}_S(K_1, \kappa)$ be the functions dual to $\Psi(0, \dots, 2, \dots, 0)$ (the 2 in the i th place) and $\Psi(0, \dots, 1, \dots, 0)$ (the 1 in the i th place) respectively. The following conclusion is the Theorem 4.1 in [23].

Lemma 3.2. *The algebra $H^*(S, \kappa)$ is generated by ξ_i, η_i ($1 \leq i \leq N$) with $\deg \xi_i = 2$ and $\deg \eta_i = 1$, subject to the relations*

$$\xi_i \xi_j = q_{ij}^{N_i N_j} \xi_j \xi_i, \quad \eta_i \xi_j = q_{ji}^{N_j} \xi_j \eta_i, \quad \eta_i \eta_j = -q_{ji} \eta_j \eta_i$$

where $q_{ij} = q_{ji}^{-1}$ if $i > j$.

For any two nonnegative integers m, n , define an algebra $\wedge(m|n)$ as follows. It is generated by $\eta_1, \dots, \eta_{m+n}$ with relations

$$\eta_i \eta_j = \begin{cases} \eta_j \eta_i, & 1 \leq i < j \leq m \\ -\eta_j \eta_i, & 1 \leq i < j \text{ and } j > m, \end{cases} \quad \eta_i^2 = 0.$$

Proposition 3.3. *Let \mathfrak{g} be a basic classical Lie superalgebra different from $\mathbf{A}(1, 1)$ and Φ its root supersystem. Then*

$$H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa) \cong \kappa[\xi_1, \dots, \xi_{m+n}] \otimes \wedge(m|n)$$

where $m = \Phi_1^\#$, $n = \Phi_0^\# - \Phi_{12}^\#$ and $\deg \xi_i = 2$, $\deg \eta_i = 1$.

Proof. It is a direct consequence of Lemma 3.2 and the definition of $|\text{Gr}^2(R_{\mathfrak{g}})|$. \square

Proposition 3.4. *Let \mathfrak{g} be a basic classical Lie superalgebra different from $\mathbf{A}(1, 1)$. Fix notions as above. Then*

(1) $H^*(|\text{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa) \cong H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa)^{\mathbf{u}(\mathfrak{h})}$ where the action of $\mathbf{u}(\mathfrak{h})$ on $H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ is given through

$$(3.3) \quad h \cdot \xi_i = -N_i \alpha_i(h) \xi_i, \quad h \cdot \eta_i = -\alpha_i(h) \eta_i,$$

for $1 \leq i \leq \Phi^\# - \Phi_{12}^\#$ and $h \in \mathbf{u}(\mathfrak{h})$.

(2) $H^*(\text{Gr}^2(\mathbf{u}(\mathfrak{g})), \kappa) \cong H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa)^{\mathbf{u}(\mathfrak{h}) \otimes \kappa \mathbb{Z}_2}$ where the action of $\kappa \mathbb{Z}_2 = \kappa \langle g | g^2 = 1 \rangle$ on $H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ is given through

$$(3.4) \quad g \cdot \xi_i = \xi_i, \quad g \cdot \eta_i = \begin{cases} -\eta_i, & i \leq \Phi_1^\# \\ \eta_i, & i > \Phi_1^\#. \end{cases}$$

Proof. (1) To give the action $\mathbf{u}(\mathfrak{h})$ on $H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa)$, we explain ξ_i, η_i and $h \in \mathbf{u}(\mathfrak{h})$ as chain maps $K_\bullet \rightarrow K_\bullet$. Then action is given by forming the

commutators of compositions of these chain maps. In fact, ξ_i, η_i has been explained as chain maps in [23] and they are described as follows:

$$\xi_i(\Psi(a_1, \dots, a_N)) = \prod_{i < l} q_{il}^{N_i \tau_l(a_l)} \Psi(a_1, \dots, a_i - 2, \dots, a_N),$$

$$\eta_i(\Psi(a_1, \dots, a_N)) = c x_i^{\sigma_i(a_i) - 1} \Psi(a_1, \dots, a_i - 1, \dots, a_N)$$

where $c = \prod_{i < l} q_{li}^{(\sigma_i(a_i) - 1) \tau_l(a_l)} \prod_{i < l} (-1)^{a_l} q_{il}^{\tau_l(a_l)}$ and $N = \Phi^\# - \Phi_{12}^\#$. Now let h be an element in $\mathbf{u}(\mathfrak{h})$. Then $h \cdot \Psi(0, \dots, 1, \dots, 0)$ (the 1 is in the i th place) should equal to $\alpha_i(h) \Psi(0, \dots, 1, \dots, 0)$ (since one can regard $\Psi(0, \dots, 1, \dots, 0)$ as the generator x_{α_i}). Extend it to higher items and one can verify directly the following extension of $\mathbf{u}(\mathfrak{h})$ on K_\bullet indeed commutes with the differentials:

$$h \cdot \Psi(a_1, \dots, a_N) = \sum_{l=1}^N \tau_l(a_l) \alpha_l(h) \Psi(a_1, \dots, a_N)$$

for $h \in \mathbf{u}(\mathfrak{h})$ and $a_1, \dots, a_N \geq 0$. Then the induced action of $\mathbf{u}(\mathfrak{h})$ on generators ξ_i, η_i is given by

$$h \cdot \xi_i = h \xi_i - \xi_i h = -N_i \alpha_i(h) \xi_i, \quad h \cdot \eta_i = h \eta_i - \eta_i h = -\alpha_i(h) \eta_i$$

for $h \in \mathbf{u}(\mathfrak{h})$.

As $\mathbf{u}(\mathfrak{h})$ is a commutative semisimple algebra, we indeed have

$$\text{Ext}_{|\text{Gr}^2(\mathbf{u}(\mathfrak{g}))|}^i(\kappa, \kappa) = \text{Ext}_{|\text{Gr}^2(R_{\mathfrak{g}})| \# \mathbf{u}(\mathfrak{h})}^i(\kappa, \kappa) \cong \text{Ext}_{|\text{Gr}^2(R_{\mathfrak{g}})|}^i(\kappa, \kappa)^{\mathbf{u}(\mathfrak{h})}$$

for $i \geq 0$ (one can prove this fact similarly by applying the methods used in the proof of Lemma 2.7). Thus $H^*(|\text{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa) \cong H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa)^{\mathbf{u}(\mathfrak{h})}$ now.

(2) By Lemma 2.7 and (1),

$$H^*(\text{Gr}^2(\mathbf{u}(\mathfrak{g})), \kappa) \cong H^*(|\text{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa)^{\kappa \mathbb{Z}_2} \cong H^*(|\text{Gr}^2(R_{\mathfrak{g}})|, \kappa)^{\mathbf{u}(\mathfrak{h}) \otimes \kappa \mathbb{Z}_2}.$$

Similar to (1), the following action of $\kappa \mathbb{Z}_2$ on K_\bullet commutes with the differentials:

$$g \cdot \Psi(a_1, \dots, a_N) = \prod_{l=1}^{\Phi_1^\#} (-1)^{\tau_l(a_l)} \Psi(a_1, \dots, a_N).$$

This induces the action

$$g \cdot \xi_i = g \xi_i g^{-1} = \begin{cases} (-1)^{N_i} \xi_i, & i \leq \Phi_1^\# \\ \xi_i, & i > \Phi_1^\# \end{cases}, \quad g \cdot \eta_i = g \eta_i g^{-1} = \begin{cases} -\eta_i, & i \leq \Phi_1^\# \\ \eta_i, & i > \Phi_1^\# \end{cases}.$$

By the definition of N_i in (3.1), it is an even when $i \leq \Phi_1^\#$. \square

3.2. Cohomology of $\text{Gr}^1(\mathfrak{u}(\mathfrak{g}))$. For a basic classical Lie superalgebra \mathfrak{g} , its enveloping algebra is denoted by $U(\mathfrak{g})$. As the case of $\mathfrak{u}(\mathfrak{g})$, define

$$\deg_1(\mathfrak{h}) := 0, \quad \deg_1\left(\bigoplus_{\alpha \in \Phi_1} \mathfrak{g}_\alpha\right) := 1, \quad \deg_1\left(\bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha\right) := 2.$$

Then we will get a filtration on $U(\mathfrak{g})$ and associated graded algebra

$$\text{Gr}^1(U(\mathfrak{g})) = \tilde{R}_{\mathfrak{g}} \# U(\mathfrak{h})$$

similarly.

Lemma 3.5. *Assume that $\mathfrak{g} \neq \mathbf{A}(1,1)$ and let x_α defined as in Lemma 2.11. Then the graded algebra $\tilde{R}_{\mathfrak{g}}$ has the following PBW basis consisting of elements*

$$(3.5) \quad x_{\alpha_1}^{a_1} \cdots x_{\alpha_r}^{a_r} x_{\beta_1}^{b_1} \cdots x_{\beta_s}^{b_s} x_{\gamma_1}^{c_1} \cdots x_{\gamma_t}^{c_t}$$

where $\alpha_i \in \Phi_{11}, \beta_j \in \Phi_{12}, \gamma_k \in \Phi_0$, $r = \Phi_{11}^\#, s = \Phi_{12}^\#, t = \Phi_0^\# - \Phi_{12}^\#$ and $0 \leq a_i < 2, b_j, c_k \in \mathbb{N}$ for $1 \leq i \leq r, 1 \leq j \leq s, 1 \leq k \leq t$.

Proof. Similar to that of Lemma 2.11. \square

Also, we set $\alpha_{r+i} := \beta_i$ ($1 \leq i \leq s$) and $\alpha_{r+s+i} := \gamma_i$ ($1 \leq i \leq t$) for convenience and consistence. Clearly,

$$R_{\mathfrak{g}} \cong \tilde{R}_{\mathfrak{g}} / (x_{\alpha_i}^{N_i}, 1 \leq i \leq \Phi^\# - \Phi_{12}^\#)$$

where N_i is defined the same as in (2.5). Define $N := \Phi^\# - \Phi_{12}^\#$ and for any $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{N}^N$ with $0 \leq a_i < 2$ ($1 \leq i \leq \Phi_{11}^\#$), denote the corresponding PBW basis element $x_{\alpha_1}^{a_1} \cdots x_{\alpha_N}^{a_N}$ by $\mathbf{x}^{\mathbf{a}}$ for short.

Our next aim is to give some elements of $H^2(|R_{\mathfrak{g}}|, \kappa)$. Recall $|\tilde{R}_{\mathfrak{g}}|^+$ is the augmentation ideal of $|\tilde{R}_{\mathfrak{g}}|$. Now for each $i \in \{1, \dots, N\}$, define $\tilde{\xi}_{\alpha_i} : |\tilde{R}_{\mathfrak{g}}|^+ \otimes |\tilde{R}_{\mathfrak{g}}|^+ \rightarrow \kappa$ by

$$\tilde{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}) = c_{\alpha_i}$$

where c_{α_i} is the coefficient of $x_{\alpha_i}^{N_i}$ in the product $\mathbf{x}^{\mathbf{a}} \mathbf{x}^{\mathbf{b}}$ as a linear combination of PBW basis elements. By its definition, $\tilde{\xi}_{\alpha_i}$ is associative on $|\tilde{R}_{\mathfrak{g}}|^+$ and thus it may be extended to a normalized two-cocycle on $|\tilde{R}_{\mathfrak{g}}|$. We next show that $\tilde{\xi}_{\alpha_i}$ factors through the quotient map $\pi : |\tilde{R}_{\mathfrak{g}}| \rightarrow |R_{\mathfrak{g}}|$ to give a nonzero two-cocycle on $|R_{\mathfrak{g}}|$. To attack this, we need show the $\tilde{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}) = 0$ whenever $\mathbf{x}^{\mathbf{a}}$ or $\mathbf{x}^{\mathbf{b}}$ is in the kernel of the quotient map π . Suppose $\mathbf{x}^{\mathbf{a}} \in \text{Ker } \pi$, which implies that $a_j \geq N_j$ for some j . By the proof of Lemma 2.10, $x_{\alpha_j}^{N_j}$ lies in the center of $U(\mathfrak{g})$ and so $\mathbf{x}^{\mathbf{a}} = x_{\alpha_j}^{N_j} \mathbf{x}^{\mathbf{c}}$ for some $\mathbf{c} \in \mathbb{N}^N$. Then $\tilde{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}) = \tilde{\xi}_{\alpha_i}(x_{\alpha_j}^{N_j} \mathbf{x}^{\mathbf{c}}, \mathbf{x}^{\mathbf{b}})$ is the coefficient of $x_{\alpha_i}^{N_i}$ in the product $x_{\alpha_j}^{N_j} \mathbf{x}^{\mathbf{c}} \mathbf{x}^{\mathbf{b}}$. It is zero now: If $j = i$, then since $\mathbf{x}^{\mathbf{b}} \in |\tilde{R}_{\mathfrak{g}}|^+$, this product cannot have a nonzero coefficient for $x_{\alpha_i}^{N_i}$. If $j \neq i$, the same conclusion is

true since $x_{\alpha_j}^{N_j}$ is always a factor of $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}$. One can show the result similarly in the case $\mathbf{x}^{\mathbf{b}} \in \text{Ker } \pi$.

Choose the section $\tilde{\sim} : |R_{\mathfrak{g}}| \rightarrow |\tilde{R}_{\mathfrak{g}}|$ of π which just sent the PBW basis elements in $R_{\mathfrak{g}}$, given in Lemma 2.11, to the same elements in $\tilde{R}_{\mathfrak{g}}$, described in Lemma 3.5. Since $\tilde{\xi}_{\alpha_i}$ factors through $\pi : |\tilde{R}_{\mathfrak{g}}| \rightarrow |R_{\mathfrak{g}}|$, we may define $\hat{\xi}_{\alpha_i} : |R_{\mathfrak{g}}|^+ \otimes |R_{\mathfrak{g}}|^+ \rightarrow \kappa$ by

$$\hat{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}) := \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}, \tilde{\mathbf{x}}^{\mathbf{b}})$$

where $\tilde{\mathbf{x}}^{\mathbf{a}}, \tilde{\mathbf{x}}^{\mathbf{b}}$ are defined via the section $\tilde{\sim}$.

Proposition 3.6. *The set $\{\hat{\xi}_{\alpha_i} | i = 1, \dots, N\}$ represents a linear independent subset of $H^2(|R_{\mathfrak{g}}|, \kappa)$.*

Proof. At first, let us show that every $\hat{\xi}_{\alpha_i}$ is a 2-cocycle. For this, it is enough to show that it is associative, that is, for any three PBW basis elements $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}}$, we have $\hat{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}}) = \hat{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}})$. Since π is an algebra homomorphism, we have $\tilde{\mathbf{x}}^{\mathbf{a}}\tilde{\mathbf{x}}^{\mathbf{b}} = \widetilde{\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}} + y$ and $\tilde{\mathbf{x}}^{\mathbf{b}}\tilde{\mathbf{x}}^{\mathbf{c}} = \widetilde{\mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}} + z$ for $y, z \in \text{Ker } \pi$. Therefore,

$$\begin{aligned} \hat{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}}) &= \tilde{\xi}_{\alpha_i}(\widetilde{\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}}, \tilde{\mathbf{x}}^{\mathbf{c}}) \\ &= \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}\tilde{\mathbf{x}}^{\mathbf{b}} - y, \tilde{\mathbf{x}}^{\mathbf{c}}) = \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}\tilde{\mathbf{x}}^{\mathbf{b}}, \tilde{\mathbf{x}}^{\mathbf{c}}) \\ &= \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}, \tilde{\mathbf{x}}^{\mathbf{b}}\tilde{\mathbf{x}}^{\mathbf{c}}) \\ &= \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}, \widetilde{\mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}} + z) = \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}, \widetilde{\mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}}) \\ &= \hat{\xi}_{\alpha_i}(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}). \end{aligned}$$

Next, let us show that they are linear independent in $H^2(|R_{\mathfrak{g}}|, \kappa)$. It is equivalent to show that for any linear combination $f = \sum_{i=1}^N c_i \hat{\xi}_{\alpha_i}$, if it is a coboundary then every $c_i = 0$. Assume that $f = \partial h$ for some $h : |R_{\mathfrak{g}}|^+ \rightarrow \kappa$. Then

$$c_i = f(x_{\alpha_i}, x_{\alpha_i}^{N_i-1}) = \partial h(x_{\alpha_i}, x_{\alpha_i}^{N_i-1}) = -h(x_{\alpha_i}^{N_i}) = 0$$

since $x_{\alpha_i}^{N_i} = 0$ in $|R_{\mathfrak{g}}|$ by Lemma 2.10. \square

See Section 6 in [24] for the definitions of such elements in the case of pointed Hopf algebras. We are now in the position to prove the following theorem.

Theorem 3.7. *The algebra $H^*(|\text{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$ is finitely generated. If M is a finitely generated $|\text{Gr}^1(\mathbf{u}(\mathfrak{g}))|$ -module, then $H^*(|\text{Gr}^1(\mathbf{u}(\mathfrak{g}))|, M)$ is a finitely generated module over $H^*(|\text{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$.*

Proof. By Lemma 2.12, there is a filtration on $|R_{\mathfrak{g}}|$ and results a graded algebra $|\text{Gr}^2(R_{\mathfrak{g}})|$. As the filtration is finite, there is a convergent spectral sequence associated to the filtration by 5.4.1 in [35]:

$$(3.6) \quad E_1^{s,t} = H^{s+t}(\text{Gr}_{(s)}^2(|R_{\mathfrak{g}}|), \kappa) \Rightarrow H^{s+t}(|R_{\mathfrak{g}}|, \kappa).$$

Since the PBW basis elements (2.4) are eigenvectors for $\mathbf{u}(\mathfrak{h})$, the action of $\mathbf{u}(\mathfrak{h})$ on $|R_{\mathfrak{g}}|$ preserves the filtration and we further get a spectral sequence converging to the cohomology of $|R_{\mathfrak{g}}\#\mathbf{u}(\mathfrak{h})| = |\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|$:

$$(3.7) \quad H^{s+t}(\mathrm{Gr}_{(s)}^2(|R_{\mathfrak{g}}|), \kappa)^{\mathbf{u}(\mathfrak{h})} \Rightarrow H^{s+t}(|R_{\mathfrak{g}}|, \kappa)^{\mathbf{u}(\mathfrak{h})} \cong H^{s+t}(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa),$$

where the isomorphism “ \cong ” can be proved similarly just as in the proof of Proposition 3.4. We can replace κ by M in (3.6), (3.7) to get convergent spectral sequences with coefficients in M .

By Proposition 3.6, we have some elements $\hat{\xi}_{\alpha_i}$ in $H^2(|R_{\mathfrak{g}}|, \kappa)$. We wish to related the functions $\hat{\xi}_{\alpha_i}$ to elements on the E_1 -page of the spectral sequence (3.6). In fact, one can copy the arguments stating before Lemma 5.1 in [23] and can assume that $\hat{\xi}_{\alpha_i} \in E_1^{c, 2-c} \cong H^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ for some $c \in \mathbb{N}$. Since $\hat{\xi}_{\alpha_i} \in H^2(|R_{\mathfrak{g}}|, \kappa)$, they are permanent cycles. Now, by Proposition 3.3, $H^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ is indeed spanned by ξ_i for $1 \leq i \leq N = \Phi^\# - \Phi_{12}^\#$.

Claim 1. *In $H^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$, $\xi_i = \hat{\xi}_{\alpha_i}$. (The proof of this claim is the same with that of Lemma 5.1 in [23] and thus is omitted.)*

Let $B^{*,*}$ be the bigraded subalgebra of $E_1^{*,*}$ generated by the elements ξ_i . By the claim 1, $B^{*,*}$ consists of permanent cycles. Let $A^{*,*}$ be the subalgebra of $B^{*,*}$ generated by ξ_i^p where p is the characteristic of κ . By (3.3) and (3.4) in Proposition 3.4, ξ_i^p is invariant under the action of $\mathbf{u}(\mathfrak{h}) \otimes \kappa\mathbb{Z}_2$. Therefore, $A^{*,*}$ is a subalgebra of $H^*(\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g})), \kappa)$. Lemma 2.1 implies that $A^{*,*}$ is commutative since it is concentrated in even (total) degrees.

Claim 2. *$A^{*,*}$ satisfies the hypotheses of Lemma 2.13. To show it, it is enough to show that $E_1^{*,*}$ is a finitely generated module over $A^{*,*}$. Proposition 3.3 implies $E_1^{*,*} \cong H^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ is generated by ξ_i and η_i where $\eta_i^2 = 0$. Hence $E_1^{*,*}$ is a finitely generated module over $B^{*,*}$ which is clearly a finitely generated module over $A^{*,*}$. Therefore, the claim is proved.*

Thus Lemma 2.13 (1) is applied and so $H^*(|R_{\mathfrak{g}}|, \kappa)$ is a Noetherian $\mathrm{Tot}(A^{*,*})$ -module. Moreover, the action of $\mathbf{u}(\mathfrak{h})$ on $H^*(|R_{\mathfrak{g}}|, \kappa)$ is compatible with the action on $A^{*,*}$, since the spectral sequence (3.6) is compatible with the action of $\mathbf{u}(\mathfrak{h})$. Therefore, $H^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa) \cong H^*(|R_{\mathfrak{g}}|, \kappa)^{\mathbf{u}(\mathfrak{h})}$ is a Noetherian $\mathrm{Tot}(A^{*,*})$ -module. Now, $\mathrm{Tot}(A^{*,*})$ is finitely generated since $A^{*,*}$ is just the polynomial algebra generated by ξ_i^p . We conclude that $H^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$ is finitely generated.

The second statement of the this theorem follows by a direct application of Lemma 2.13 (2). \square

Next result is a direct consequence of Theorem 3.7 and Lemma 2.7.

Corollary 3.8. *The algebra $H^*(\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g})), \kappa)$ is finitely generated. If M is a finitely generated $\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))$ -supermodule, then $H^*(\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g})), M)$ is a finitely generated module over $H^*(\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g})), \kappa)$.*

3.3. Cohomology of $\mathfrak{u}(\mathfrak{g})$. In this subsection, we will give the proof of Theorem 3.1 provided $\mathfrak{g} \neq \mathbf{A}(1,1)$. Similar to Subsection 3.2, we have convergent spectral sequences associated the first kind of filtration given before Lemma 2.11:

$$(3.8) \quad E_1^{s,t} = H^{s+t}(\mathrm{Gr}_{(s)}^1(|\mathfrak{u}(\mathfrak{g})|), \kappa) \Rightarrow H^{s+t}(|\mathfrak{u}(\mathfrak{g})|, \kappa),$$

$$(3.9) \quad H^{s+t}(\mathrm{Gr}_{(s)}^1(|\mathfrak{u}(\mathfrak{g})|), M) \Rightarrow H^{s+t}(|\mathfrak{u}(\mathfrak{g})|, M),$$

for any $|\mathfrak{u}(\mathfrak{g})|$ -module M .

Previously, we identify the element $\xi_i \in H^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ with the element $\hat{\xi}_{\alpha_i} \in H^2(|R_{\mathfrak{g}}|, \kappa)$. From this, we know that ξ_i is a permanent cycle and $H^*(|R_{\mathfrak{g}} \# \mathfrak{u}(\mathfrak{h})|, \kappa) = H^*(|\mathrm{Gr}^1(\mathfrak{u}(\mathfrak{g}))|, \kappa)$ is finitely generated over the subalgebra generated by all $\hat{\xi}_{\alpha_i}^p$. So our next aim is to find an element $f_{\alpha_i} \in H^*(|\mathfrak{u}(\mathfrak{g})|, \kappa)$ which can be identified with $\hat{\xi}_{\alpha_i}^p$. If so, $\hat{\xi}_{\alpha_i}^p$ will be permanent cycles and Lemma 2.13 can be applied.

For each $i \in \{1, \dots, \Phi^\# - \Phi_{12}^\#\}$, let α_i be the corresponding root. For our purpose, we choose a PBW basis of $U(\mathfrak{g})$, described as in Proposition 2.5 (1), with requirements: $s_1 = \Phi_{11}^\#, s = \Phi_1^\#, x_i = x_{\alpha_i}$ for $1 \leq i \leq s$ and $y_j = x_{\alpha_{s+j}}$ for $1 \leq j \leq \Phi_0^\# - \Phi_{12}^\#$ where x_{α_k} is defined before Lemma 2.11. Roughly speaking, we just want the PBW basis elements given in Lemma 3.5 are still PBW basis elements in the following discussions. We choose a PBW basis for $\mathfrak{u}(\mathfrak{g})$ with the same requirements as $U(\mathfrak{g})$. Such PBW basis will be fixed from now on until the end of this subsection.

Define a κ -linear function $\tilde{f}_{\alpha_i} : (|U(\mathfrak{g})|^+)^{2p} \rightarrow \kappa$ as follows. Let r_1, \dots, r_{2p} be PBW basis elements. If all of them have no factors belonging to $U(\mathfrak{h})$, then

$$\tilde{f}_{\alpha_i}(r_1 \otimes \dots \otimes r_{2p}) := c_{12}c_{34} \dots c_{2p-1,2p}$$

where c_{ij} is the coefficient of $x_{\alpha_i}^{N_i}$ in the product $r_i r_j$ as a linear combination of PBW basis elements. And set \tilde{f}_{α_i} to be zero whenever there is a r_i which contains a factor living in $U(\mathfrak{h})$.

Similar to Subsection 3.2, we will show that \tilde{f}_{α_i} factors through the quotient $\pi : U(\mathfrak{g}) \rightarrow \mathfrak{u}(\mathfrak{g})$ to give a map $(|\mathfrak{u}(\mathfrak{g})|^+)^{2p} \rightarrow \kappa$. Note that by the definition of \tilde{f}_{α_i} , it is always 0 whenever the elements of $U(\mathfrak{h})$ appear in a PBW basis element. So we need only to consider the PBW basis elements totally the same as that of $\tilde{R}_{\mathfrak{g}}$. So we can apply the same arguments designed for $\tilde{\xi}_{\alpha_i}$ to \tilde{f}_{α_i} and show that it indeed factors through the quotient map $\pi : U(\mathfrak{g}) \rightarrow \mathfrak{u}(\mathfrak{g})$. Also, we choose a section $\tilde{\sim} : \mathfrak{u}(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ of the quotient map π . Then define $f_{\alpha_i} : (|\mathfrak{u}(\mathfrak{g})|^+)^{2p} \rightarrow \kappa$ by setting

$$f_{\alpha_i}(r_1 \otimes \dots \otimes r_{2p}) := \tilde{f}_{\alpha_i}(\tilde{r}_1 \otimes \dots \otimes \tilde{r}_{2p})$$

for PBW basis elements $r_1, \dots, r_{2p} \in \mathfrak{u}(\mathfrak{g})$.

Proposition 3.9. *The set $\{f_{\alpha_i} | i = 1, \dots, \Phi^\# - \Phi_{12}^\#\}$ represents a linear independent subset of $H^{2p}(|\mathbf{u}(\mathfrak{g})|, \kappa)$.*

Proof. The proof is similar to that of Lemma 6.2 in [23] and Proposition 3.6. For completeness, we write it out.

Firstly, we show that \tilde{f}_{α_i} is a $2p$ -cocycle on $|U(\mathfrak{g})|$. Let $r_0, \dots, r_{2p} \in |U(\mathfrak{g})|^+$ be PBW basis elements without factors coming from $U(\mathfrak{h})$. Then

$$\partial(\tilde{f}_{\alpha_i})(r_0 \otimes \dots \otimes r_{2p}) = \sum_{j=0}^{2p-1} (-1)^{i+1} \tilde{f}_{\alpha_i}(r_0 \otimes \dots \otimes r_j r_{j+1} \otimes \dots \otimes r_{2p}).$$

By the definition of \tilde{f}_{α_i} , the first two terms cancel and similarly for all other terms. So $\partial(\tilde{f}_{\alpha_i}) = 0$.

Now we verify that f_{α_i} is a $2p$ -cocycle. Also, let $r_0, \dots, r_{2p} \in |\mathbf{u}(\mathfrak{g})|^+$ be PBW basis elements. Then

$$\partial(f_{\alpha_i})(r_0 \otimes \dots \otimes r_{2p}) = \sum_{j=0}^{2p-1} (-1)^{i+1} f_{\alpha_i}(r_0 \otimes \dots \otimes r_j r_{j+1} \otimes \dots \otimes r_{2p}).$$

Using the same methods as in the proof of Proposition 3.6, we have

$$\begin{aligned} f_{\alpha_i}(r_0 r_1 \otimes r_2 \otimes \dots \otimes r_{2p}) &= \tilde{f}_{\alpha_i}(\widetilde{r_0 r_1} \otimes \tilde{r}_2 \otimes \dots \otimes \tilde{r}_{2p}) \\ &= \tilde{f}_{\alpha_i}(\tilde{r}_0 \tilde{r}_1 \otimes \tilde{r}_2 \otimes \dots \otimes \tilde{r}_{2p}) \\ &= \tilde{f}_{\alpha_i}(\tilde{r}_0 \otimes \tilde{r}_1 \tilde{r}_2 \otimes \dots \otimes \tilde{r}_{2p}) \\ &= \tilde{f}_{\alpha_i}(\tilde{r}_0 \otimes \widetilde{r_1 r_2} \otimes \dots \otimes \tilde{r}_{2p}) \\ &= f_{\alpha_i}(r_0 \otimes r_1 r_2 \otimes \dots \otimes r_{2p}). \end{aligned}$$

Similarly, we have

$$f_{\alpha_i}(r_0 \otimes \dots \otimes r_j r_{j+1} \otimes \dots \otimes r_{2p}) = f_{\alpha_i}(r_0 \otimes \dots \otimes r_{j+1} r_{j+2} \otimes \dots \otimes r_{2p})$$

for $j = 0, \dots, 2p - 2$. So $\partial(f_{\alpha_i}) = 0$.

Now assume that $\sum_i c_i f_{\alpha_i} = \partial h$ for some $h \in \text{Hom}_\kappa((|\mathbf{u}(\mathfrak{g})|^+)^{\otimes 2p-1}, \kappa)$. Then for each i ,

$$\begin{aligned} c_i &= \left(\sum_j c_j f_{\alpha_j} \right) (x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1} \otimes \dots \otimes x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1}) \\ &= (\partial h)(x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1} \otimes \dots \otimes x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1}) \\ &= \sum \pm h(x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1} \otimes \dots \otimes x_{\alpha_i}^{N_i} \otimes \dots \otimes x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1}) \\ &= 0 \end{aligned}$$

since $x_{\alpha_i}^{N_i} = 0$ in $\mathbf{u}(\mathfrak{g})$ by Lemma 2.10.

□

Proof of Theorem 3.1 in case $\mathfrak{g} \neq \mathbf{A}(1, 1)$. The functions f_{α_i} correspond to their counterpart $\hat{\xi}_{\alpha_i}^p$ defined on $|\text{Gr}^1(\mathbf{u}(\mathfrak{g}))|$, in the E_1 -page of the spectral

sequence (3.8), by observing that they are the same functions at the level of chain complex (2.3) where we need replace $|\mathbf{u}(\mathfrak{g})|^+$ by $|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|^+$. Thus Proposition 3.9 implies that the function $\hat{\xi}_{\alpha_i}^p$ is a permanent cycle. Now we have known that $E_1^{*,*} \cong H^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$ is finitely generated over the subalgebra $A^{*,*}$ generated by all $\hat{\xi}_{\alpha_i}^p$ (see the proof of Theorem 3.7). Thus $A^{*,*}$ satisfies the conditions of Lemma 2.13 and thus $H^*(|\mathbf{u}(\mathfrak{g})|, \kappa)$ is a Noetherian $\mathrm{Tot}(A^{*,*})$ -module and thus finitely generated. By Lemma 2.7, the first part of Theorem 3.1 is proved. The second part can be prove similarly by applying Lemma 2.13 (2) and Lemma 2.7.

3.4. The case $\mathfrak{g} = \mathbf{A}(1, 1)$. We deal with the case $\mathfrak{g} = \mathbf{A}(1, 1)$ in a bigger context: Those basic classical Lie superalgebras with Φ_{12} being empty. By the descriptions of root supersystems given in Section 2.5.4 in [18], this includes all basic classical Lie superalgebras except $\mathbf{B}(m, n)$ and $\mathbf{G}(3)$. For such Lie superalgebras, we have a nice filtration on them.

We give a notion at first. For a coalgebra C and $D \subseteq C$ a subcoalgebra of C , define

$$\wedge^0 D := D, \quad \wedge^1 D := \Delta^{-1}(C \otimes D + D \otimes C),$$

$$\wedge^i D := \wedge^1(\wedge^{i-1} D) = \Delta^{-1}(C \otimes \wedge^{i-1} D + \wedge^{i-1} D \otimes C)$$

for $i \geq 2$. If D contains the coradical C_0 of C , by definition C_0 is the sum of all simple subcoalgebras of C , then $D \subseteq \wedge D \subseteq \wedge^2 D \subseteq \dots$ will give a filtration of C . See Chapter 5 in [25] for details.

Let \mathfrak{g} be a basic classical Lie superalgebra with $\Phi_{12} = \emptyset$. Then $\mathbf{u}(\mathfrak{g})$ is a finite dimensional super cocommutative Hopf algebra and its coradical is κ . Define

$$F^i \mathbf{u}(\mathfrak{g}) := \wedge^i \mathbf{u}(\mathfrak{h})$$

for $i \geq 0$ and then this gives a filtration of $\mathbf{u}(\mathfrak{g})$. The associated graded algebra is denoted by $\mathrm{gr}(\mathbf{u}(\mathfrak{g}))$. It is a superalgebra naturally. For any $\alpha \in \Phi$, we fix a basis b_α of \mathfrak{g}_α . By taking the union of such b_α , we get a basis of $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Denote this basis by $\{x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n}\}$ and assume that $x_i \in \bigoplus_{\alpha \in \Phi_1} \mathfrak{g}_\alpha$ for $1 \leq i \leq m$ while $x_i \notin \bigoplus_{\alpha \in \Phi_1} \mathfrak{g}_\alpha$ for $m < i \leq m+n$.

Lemma 3.10. $\mathrm{gr}(\mathbf{u}(\mathfrak{g})) \cong S_{\mathfrak{g}} \# \mathbf{u}(\mathfrak{h})$ where $S_{\mathfrak{g}}$ is generated by x_1, \dots, x_{m+n} with relations

$$(3.10) \quad x_i x_j = \begin{cases} -x_j x_i, & 1 \leq i < j \leq m \\ x_j x_i, & 1 \leq i < j, j > m, \end{cases} \quad x_i^{n_i} = 0,$$

$$\text{where } n_i = \begin{cases} 2, & 1 \leq i \leq m \\ p, & m < i \leq m+n. \end{cases}$$

Proof. Here the action of $\mathbf{u}(\mathfrak{h})$ on $S_{\mathfrak{g}}$ is gotten through extending the actions of \mathfrak{h} on $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$ naturally. By the definition of the coproduct of $\mathbf{u}(\mathfrak{g})$, $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha} \subset \wedge^1 \mathbf{u}(\mathfrak{h})$. So $[x_i, x_j] \in \wedge^1 \mathbf{u}(\mathfrak{h})$. This implies we have

$$[x_i, x_j] = 0$$

in $\text{gr}(\mathbf{u}(\mathfrak{g}))$. It is direct to show that every $x_i^{n_i}$ is still a primitive element and so $x_i^{n_i} \in \wedge^1 \mathbf{u}(\mathfrak{h})$ too. Therefore, $x_i^{n_i} = 0$ in $\text{gr}(\mathbf{u}(\mathfrak{g}))$. Now all relations in (3.10) are fulfilled. By comparing the dimensions, we indeed get the desire isomorphism. \square

Proof of Theorem 3.1 in case $\Phi_{12} = \phi$. Since $\mathbf{u}(\mathfrak{g})$ is finite dimensional, then the filtration $F^0 \mathbf{u}(\mathfrak{g}) \subset F^1 \mathbf{u}(\mathfrak{g}) \subset \dots$ is finite, that is, there is $n \in \mathbb{N}$ such that $F^n \mathbf{u}(\mathfrak{g}) = \mathbf{u}(\mathfrak{g})$. So have a convergent spectral sequence

$$(3.11) \quad E_1^{s,t} = H^{s+t}(\text{gr}_{(s)}(|\mathbf{u}(\mathfrak{g})|), \kappa) \Rightarrow H^{s+t}(|\mathbf{u}(\mathfrak{g})|, \kappa).$$

By Lemma 3.10, $|\text{gr} \mathbf{u}(\mathfrak{g})| \cong |S_{\mathfrak{g}} \# \mathbf{u}(\mathfrak{h})|$. Now it is clear $|S_{\mathfrak{g}}|$ is a quantum complete intersection algebra (see the second paragraph of Subsection 3.1). Thus its cohomology algebra is clear by Lemma 3.2. Actually, similar to Proposition 3.3, we have

$$H^*(|S_{\mathfrak{g}}|, \kappa) \cong k[\xi_1, \dots, \xi_{m+n}] \otimes \wedge(m|n)$$

with $m = \Phi_1^{\#}, n = \Phi_0^{\#}$ and $\deg \xi_i = 2, \deg \eta_i = 1$. Also one can get that $\xi_i^p \in H^*(|S_{\mathfrak{g}}|, \kappa)^{\mathbf{u}(\mathfrak{h})} \cong H^*(|\text{gr}(\mathbf{u}(\mathfrak{g}))|, \kappa)$. By applying the same discussions used in the proof of Claim 2 in that of Theorem 3.7, $E_1^{*,*}$ is finitely generated over the subalgebra generated by all ξ_i^p . So Lemma 2.13 (1) can be applied if we can show all ξ_i^p are permanent cycles. In fact, we can define $f_i \in H^{2p}(|\mathbf{u}(\mathfrak{g})|, \kappa)$ through the same way as that of f_{α_i} (see Proposition 3.9) and get f_i corresponds to its counterpart ξ_i^p defined on $|\text{gr}(\mathbf{u}(\mathfrak{g}))|$. Therefore, every ξ_i^p is a permanent cycle and thus $H^*(|\mathbf{u}(\mathfrak{g})|, \kappa)$ is a finitely generated algebra. Using Lemma 2.7, we know that $H^*(\mathbf{u}(\mathfrak{g}), \kappa)$ is also finitely generated as an algebra.

The second part of the theorem can be proved by applying Lemma 2.13 (2) and Lemma 2.7 now.

Remark 3.11. To show the theorem, we cannot apply the filtration developed in this subsection to Lie superalgebras $\mathbf{B}(m, n), \mathbf{G}(3)$ directly since otherwise more nilpotent elements will be created. On the contrary, the two kinds of filtration given in Section 2 can be applied to $\mathbf{A}(1, 1)$ and indeed $\text{Gr}^2(\mathbf{A}(1, 1)) = \text{gr}(\mathbf{A}(1, 1))$. But in the case of $\mathfrak{g} = \mathbf{A}(1, 1)$, it is possible that $\dim_{\kappa} \mathfrak{g}_{\alpha} \geq 2$ and so the notation x_{α} has no meaning now. Therefore, if we want to deal with all basic classical Lie superalgebras in a unified way (that is, by using two kinds of filtration), the notations and descriptions will be too delicate to grasp the main line.

4. SUPPORT VARIETIES AND REPRESENTATION TYPE OF LIE SUPERALGEBRAS

In this section, we will recall the definition of the support variety of a module and give its relation with the complexity of this module. As a consequence, we will show that only $|\mathbf{u}(\mathfrak{sl}_2)|$, $|\mathbf{u}(\mathfrak{osp}(1|2))|$ are tame and the others $|\mathbf{u}(\mathfrak{g})|$ are all wild (see Section 5 for explicit description of $\mathfrak{osp}(1|2)$).

Let \mathfrak{g} be a basic classical Lie superalgebra and N a finitely generated left $\mathbf{u}(\mathfrak{g})$ -supermodule. By Corollary 2.8 and Theorem 3.1, $H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa)$ is a finitely generated commutative algebra and $H^*(\mathbf{u}(\mathfrak{g}), N)$ is a finitely generated $H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa)$ -module. In particular, for any finitely generated $\mathbf{u}(\mathfrak{g})$ -supermodule M , $\text{Ext}_{\mathbf{u}(\mathfrak{g})}^*(M, M) := \bigoplus_{i \geq 0} H_{\mathbf{u}(\mathfrak{g})}^i(M, M) \cong \bigoplus_{i \geq 0} H^i(\mathbf{u}(\mathfrak{g}), M^* \otimes M)$ is finitely generated over $H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa)$ where M^* is the dual $\mathbf{u}(\mathfrak{g})$ -module of M . Let I_M be the annihilator of action of $H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa)$ on $\text{Ext}_{\mathbf{u}(\mathfrak{g})}^*(M, M)$. The *cohomological support variety* of M is defined to be

$$\mathcal{V}_{\mathbf{u}(\mathfrak{g})}(M) := Z(I_M) \subset \text{Maxspec}(H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa)).$$

Note that we can regard M as a $\mathbf{u}(\mathfrak{g}) \# \kappa \mathbb{Z}_2$ -module by Lemma 2.2.

Let A be an associative algebra, M an A -module with minimal projective resolution

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Then the *complexity* of M is defined to be the integer

$$C_A(M) := \min\{c \in \mathbb{N}_0 \cup \infty \mid \exists \lambda > 0 : \dim_k P_n \leq \lambda n^{c-1}, \forall n \geq 1\}.$$

Lemma 4.1. *Let \mathfrak{g} be a basic classical Lie superalgebra and M a finitely generated left $\mathbf{u}(\mathfrak{g})$ -supermodule. Then*

$$\dim \mathcal{V}_{\mathbf{u}(\mathfrak{g})}(M) = C_{\mathbf{u}(\mathfrak{g}) \# \kappa \mathbb{Z}_2}(M).$$

Proof. By definition, $H^{ev}(\mathbf{u}(\mathfrak{g}), \kappa) = \bigoplus_{i \geq 0} \text{Ext}_{\mathbf{u}(\mathfrak{g}) \# \kappa \mathbb{Z}_2}^{2i}(\kappa, \kappa)$ and now $\mathbf{u}(\mathfrak{g}) \# \kappa \mathbb{Z}_2$ is an ordinary finite dimensional Hopf algebra. So this lemma is just a corollary of Proposition 2.3 in [15]. \square

Recall the finite dimensional associative algebras over an algebraically closed field κ can be divided into three classes (see [12]): A finite-dimensional algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A -modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are finite number of A - $\kappa[T]$ -bimodules M_i which are free of finite rank as right $\kappa[T]$ -modules such that all but a finite number of indecomposable A -modules of dimension d are isomorphic to $M_i \otimes_{\kappa[T]} \kappa[T]/(T - \lambda)$ for $\lambda \in \kappa$. We say that A is of *wild type* or A is a *wild* algebra if there is a finitely generated A - $\kappa\langle X, Y \rangle$ -bimodule B which is free as a right

$\kappa\langle X, Y \rangle$ -module such that the functor $B \otimes_{\kappa\langle X, Y \rangle} -$ from $\kappa\langle X, Y \rangle$ -mod, the category of finitely generated $\kappa\langle X, Y \rangle$ -modules, to A -mod, the category of finitely generated A -modules, preserves indecomposability and reflects isomorphisms.

The following result is a direct consequence of Proposition 3.2 in Chapter VI of [2].

Lemma 4.2. *Let A be a superalgebra and assume that characteristic of κ is not 2. Then $|A|$ and $A \# \kappa\mathbb{Z}_2$ have the same representation type.*

Remark 4.3. *For a finite dimensional superalgebra A , one also can define its representation type in the super world, that is, in the category of supermodules with even homomorphisms. By Lemma 4.2 and Lemma 2.2, the representation type of $|A|$ as an ordinary algebra is indeed the same with that of A when we consider it as a superalgebra. So to consider the representation type of a superalgebra A , it is enough to consider that of its underline algebra $|A|$.*

The following conclusion is also needed.

Lemma 4.4. *If there is a finite dimensional $\mathfrak{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ -module M such that $C_{\mathfrak{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2}(M) \geq 3$, then $\mathfrak{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ is wild.*

Proof. Let H be an arbitrary finite dimensional Hopf algebra such that $C_H(N) \geq 3$ for some H -module N . Then Theorem 3.1 in [15] implies that H is wild provided $H^*(H, \kappa)$ is finitely generated and $H^*(H, N')$ is a Noetherian module over $H^*(H, \kappa)$ for any finite dimensional H -module N' . So the lemma is proved due to our Theorem 3.1. \square

Theorem 4.5. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical Lie superalgebra over κ . Then $|\mathfrak{u}(\mathfrak{g})|$ is wild except $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{osp}(1|2)$ or $\mathfrak{g} = \mathbf{C}(2)$. Both $|\mathfrak{u}(\mathfrak{sl}_2)|$ and $|\mathfrak{u}(\mathfrak{osp}(1|2))|$ are tame.*

Proof. The proof is base on the estimation of the number $C_{\mathfrak{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2}(\kappa)$. By Proposition 2.1 in [15], we have

$$C_{\mathfrak{u}(\mathfrak{g}_0)}(\kappa) \leq C_{\mathfrak{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2}(\kappa).$$

Owing to (1.4) in [16], $\mathcal{V}_{\mathfrak{u}(\mathfrak{g}_0)}(\kappa)$ can be identified with

$$\mathcal{V}_{\mathfrak{u}(\mathfrak{g}_0)}(\kappa) := \{x \in \mathfrak{g}_0 \mid x^{[p]} = 0\} \cup \{0\}.$$

Now we have known that \mathfrak{g}_0 is a direct sum of simple Lie algebras of type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n, \mathbf{G}_2$ or κ . By Lemma 2.10,

$$\dim \mathcal{V}_{\mathfrak{u}(\mathfrak{g}_0)}(\kappa) = \dim \mathcal{V}_{\mathfrak{u}(\mathfrak{g}_0)}(\kappa) \geq 3$$

except $\mathfrak{g}_0 = \mathfrak{sl}_2$ or $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \kappa$. Thus Lemma 4.1 implies that $C_{\mathbf{u}(\mathfrak{g}) \# \kappa \mathbb{Z}_2}(\kappa) \geq C_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) = \dim \mathcal{V}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) \geq 3$ unless $\mathfrak{g}_0 = \mathfrak{sl}_2$ or $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \kappa$. The latter only appear in the case $\mathfrak{g} = \mathbf{C}(2)$. So now it is not hard to see that in the rest list of basic classical Lie superalgebras only \mathfrak{sl}_2 and $\mathfrak{osp}(1|2)$ satisfy its even part is \mathfrak{sl}_2 . By applying Lemma 4.4, the first part of theorem is proved.

For the second part, it is known that $\mathbf{u}(\mathfrak{sl}_2)$ is tame (see for example [13]). The algebra $|\mathbf{u}(\mathfrak{osp}(1|2))|$ is proved to be a tame algebra by Farnsteiner in the Example in Section 4 of [14]. \square

Conjecture 4.6. *The algebra $|\mathbf{u}(\mathbf{C}(2))|$ is a wild algebra.*

5. RESTRICTED REPRESENTATIONS OF $\mathfrak{osp}(1|2)$

Comparing with the case \mathfrak{sl}_2 , we know a little about the representations of $\mathbf{u}(\mathfrak{osp}(1|2))$. In the last section of the paper, we want to determine all finite dimensional representations of $\mathbf{u}(\mathfrak{osp}(1|2))$ inspired that fact that $|\mathbf{u}(\mathfrak{osp}(1|2))|$ is tame. To do it, the representation theory of \mathfrak{sl}_2 and the theory of Frobenius extensions are need. In this section, we only need $p \neq 2$.

5.1. \mathfrak{sl}_2 case. In this subsection, the restricted simples and projectives of \mathfrak{sl}_2 are summarized. Recall the restricted enveloping algebra $\mathbf{u}(\mathfrak{sl}_2)$ of \mathfrak{sl}_2 is generated by e, f, h with relations

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h, h^p = h, e^p = f^p = 0.$$

For any $0 \leq \lambda \leq p-1$, we define a $\lambda+1$ -dimensional $\mathbf{u}(\mathfrak{sl}_2)$ -module V_0^λ as follows. This module has a basis $v_0, v_1, \dots, v_\lambda$ and the actions of the generators are given by the following rules

$$(5.1) \quad hv_i = (\lambda - 2i)v_i, \quad ev_i = -i(\lambda + 1 - i)v_{i-1}, \quad fv_i = -v_{i+1}$$

where $i = 0, 1, \dots, \lambda$ and $v_{-1} = v_{\lambda+1} = 0$. It is well-known that $\{V_0^\lambda | 0 \leq \lambda \leq p-1\}$ forms a complete non-redundant list of simple $\mathbf{u}(\mathfrak{sl}_2)$ -modules and V_0^{p-1} is projective, which is called a *Steinberg module* in general.

It is convenient to use a graphical representation for the structures of modules. Every vertex stands for a vector from our chosen basis; arrows and dotted ones show the actions of e and f respectively. The example below is for $p = 3, \lambda = 2$.

$$\begin{array}{c} \cdot \\ \uparrow \\ \vdots \\ \downarrow \\ \cdot \\ \uparrow \\ \vdots \\ \downarrow \\ \cdot \end{array}$$

Also for any $0 \leq \lambda \leq p-2$, we define the module $P_0^{p-2-\lambda}$ by the following rules. The basis of $P_0^{p-2-\lambda}$ is $\{b_i, a_i, x_j, y_j | 0 \leq i \leq p-2-\lambda, 0 \leq j \leq \lambda\}$ and the actions of h, e, f are given by:

$$(5.2) \quad hb_i = (p-2-\lambda-2i)b_i, \quad fb_i = -b_{i+1}$$

$$(5.3) \quad eb_i = -i(p-\lambda-1-i)b_{i-1} + a_{i-1};$$

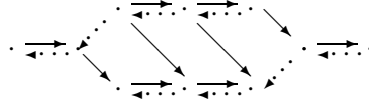
$$(5.4) \quad hy_j = (2j-\lambda)y_j, ey_j = y_{j+1}, \quad hx_j = (\lambda-2j)x_j, fx_j = -x_{j+1}$$

$$(5.5) \quad fy_j = -j(j-\lambda-1)y_{j-1}; \quad ex_j = -j(\lambda+1-j)x_{j-1};$$

$$(5.6) \quad ha_i = (p-2-\lambda-2i)a_i, \quad ea_i = -i(p-\lambda-1-i)a_{i-1}, \quad fa_i = -a_{i+1},$$

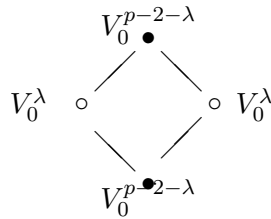
where $b_{p-1-\lambda} = y_\lambda$, $y_{\lambda+1} = a_{p-2-\lambda}$, $x_{\lambda+1} = a_0$, $b_{-1} = x_\lambda$.

The graphical description of the $P_0^{p-2-\lambda}$ (for $p=5, \lambda=1$) is indicated as follows.



It should be known that $\{P_0^{p-2-\lambda}, V_0^{p-1} | 0 \leq \lambda \leq p-2\}$ forms a complete list of indecomposable projective $\mathbf{u}(\mathfrak{sl}_2)$ -modules up to isomorphism. For safety, one also can duplicate the proof of Lemma 2.2.6 in [36] to show this fact.

One also can use the following easy way to represent the structure of $P_0^{p-2-\lambda}$ where we use \bullet or \circ to denote the composition factors of $P_0^{p-2-\lambda}$.



From this, it is not hard to see $V_0(\lambda)$ and $V_0(p-2-\lambda)$ belongs to the same block $B_0(\lambda)$ for any $0 \leq \lambda \leq p-2$ and there are exactly $\frac{p+1}{2}$ blocks. Also, one can compute the endomorphism ring $\text{End}_{\mathbf{u}(\mathfrak{sl}_2)}(P_0^\lambda \oplus P_0^{p-2-\lambda})$ out to get the basic algebra of the $B_0(\lambda)$ now. In fact, we will give such computations for $|\mathbf{u}(\mathfrak{osp}(1|2))|$ and the readers can recover the block structures of $\mathbf{u}(\mathfrak{sl}_2)$ from our computations easily.

Remark 5.1. Since the notions such as V^λ, P^λ , etc. will be used for $|\mathbf{u}(\mathfrak{osp}(1|2))|$, we add the subscript 0 to each notion and get V_0^λ, P_0^λ , etc. denoting the corresponding concepts appeared in classical case, $\mathbf{u}(\mathfrak{sl}_2)$.

5.2. Frobenius extensions. Let R be a ring and $S \subseteq R$ a subring. Suppose that α is an automorphism of S . If M is an S -module, we let ${}_{\alpha}M$ denote the S -module with a new action defined by $s * m := \alpha(s)m$. We say R is an α -Frobenius extension of S if

- (i) R is a finitely generated projective S -module, and
- (ii) there exists an isomorphism $\varphi : R \rightarrow \text{Hom}_S(R, {}_{\alpha}S)$ of (R, S) -bimodules. More on Frobenius extensions and their applications can be found in [3, 14]. For our purpose, the following several concepts are needed.

Given an endomorphism β of S , a β -associative form from R to S is a biadditive map $\langle, \rangle : R \times R \rightarrow S$ such that

$$(a) \langle sx, y \rangle = s \langle x, y \rangle, \quad (b) \langle x, ys \rangle = \langle x, y \rangle \beta(s), \quad (c) \langle xr, y \rangle = \langle x, ry \rangle$$

for all $s \in S$ and $r, x, y \in R$.

Let $\langle, \rangle : R \times R \rightarrow S$ be an α^{-1} -associative form. We say two subsets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ of R form a *dual projective pair* relative to \langle, \rangle if

$$r = \sum_{i=1}^n y_i \alpha(\langle x_i, r \rangle) = \sum_{i=1}^n \langle r, y_i \rangle x_i \quad \text{for all } r \in R.$$

Recall Theorem 1.1 in [3] states that R is an α -Frobenius extension of S if and only if there is an α^{-1} -associative form \langle, \rangle from R to S relative to which a dual projective pair $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}$ exists. Now let $R : S$ be an α -Frobenius extension and consider two R -modules M, N . Then there is a dual projective pair $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}$. The mapping $\text{Tr}_{[R:S]} : \text{Hom}_S(M, {}_{\alpha}N) \rightarrow \text{Hom}_R(M, N)$, which is defined by

$$\text{Tr}_{[R:S]}(f)(m) = \sum_{i=1}^n y_i f(x_i m), \quad \text{for } f \in \text{Hom}_S(M, {}_{\alpha}N) \text{ and } m \in M$$

is usually called the *trace map*.

The following lemma will give us a connection between $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -modules and $\mathbf{u}(\mathfrak{sl}_2)$ -modules. To describe it, we fix a notation firstly. Let R be a ring and M, N two R -modules. If M is a direct summand of N as a R -module, then we denote it by $M|N$.

Lemma 5.2. *Let M be an $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -module, then*

$$M| |\mathbf{u}(\mathfrak{osp}(1|2))| \otimes_{\mathbf{u}(\mathfrak{sl}_2)} M.$$

Proof. Define $|\mathbf{u}(\mathfrak{osp}(1|2))| \otimes_{\mathbf{u}(\mathfrak{sl}_2)} M \rightarrow M$ by $a \otimes m \mapsto am$ for $a \in |\mathbf{u}(\mathfrak{osp}(1|2))|$ and $m \in M$. Clearly, φ is an $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -epimorphism.

Recall $|\mathbf{u}(\mathfrak{osp}(1|2))| : \mathbf{u}(\mathfrak{sl}_2)$ is an *id*-Frobenius extension. Let $x = e_{21} - e_{13}, y = e_{31} + e_{12}$ where e_{ij} is the unit matrix with 1 in the i, j -entry and 0 otherwise. Then the dual projective pair is $x_1 = 1, x_2 = x, x_3 = y, x_4 =$

$xy + 1 - [x, y]$; $y_1 = xy, y_2 = y, y_3 = -x, y_4 = 1$. It is straightforward to show that $\sum_{i=1}^4 y_i x_i = 1$. For details, see the Example in page 423 of [3].

Define $\psi : M \rightarrow |\mathbf{u}(\mathfrak{osp}(1|2))| \otimes_{\mathbf{u}(\mathfrak{sl}_2)} M$ by $m \mapsto 1 \otimes m$ for $m \in M$. It is a morphism of $\mathbf{u}(\mathfrak{sl}_2)$ -modules. Therefore, the trace map $\text{Tr}_{[|\mathbf{u}(\mathfrak{osp}(1|2))| : \mathbf{u}(\mathfrak{sl}_2)]}(\psi)$ of ψ is an $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -morphism from M to $|\mathbf{u}(\mathfrak{osp}(1|2))| \otimes_{\mathbf{u}(\mathfrak{sl}_2)} M$. By definition,

$$\text{Tr}_{[|\mathbf{u}(\mathfrak{osp}(1|2))| : \mathbf{u}(\mathfrak{sl}_2)]}(\psi)(m) = \sum_{i=1}^4 y_i \otimes x_i m$$

for $m \in M$. Then $\varphi \text{Tr}_{[|\mathbf{u}(\mathfrak{osp}(1|2))| : \mathbf{u}(\mathfrak{sl}_2)]}(\psi)(m) = \sum_{i=1}^4 y_i x_i m = m$ for $m \in M$. Consequently,

$$\varphi \text{Tr}_{[|\mathbf{u}(\mathfrak{osp}(1|2))| : \mathbf{u}(\mathfrak{sl}_2)]}(\psi) = id_M$$

and thus $M[|\mathbf{u}(\mathfrak{osp}(1|2))| \otimes_{\mathbf{u}(\mathfrak{sl}_2)} M]$. \square

5.3. Simple, Projective and Blocks. In this subsection, the structures of simple modules, projective modules and the basic algebras of blocks of $|\mathbf{u}(\mathfrak{osp}(1|2))|$ are given. As a byproduct, its Auslander-Reiten quiver is determined.

5.3.1. Simple and Verma modules. As usual, for a Lie superalgebra \mathfrak{g} , its even (resp. odd) part is denoted by \mathfrak{g}_0 (resp. \mathfrak{g}_1) and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Recall that $\mathfrak{g} = \mathfrak{osp}(1|2)$ consists of 3×3 matrices in the following $(1|2)$ -block form

$$\begin{bmatrix} 0 & v & u \\ u & a & b \\ -v & c & -a \end{bmatrix}$$

for $a, b, c, u, v \in \kappa$. The even subalgebra $\mathfrak{osp}(1|2)_0$, which is isomorphic to \mathfrak{sl}_2 , is generated by

$$e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

A basis for the odd part $\mathfrak{osp}(1|2)_1$ is given by

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The commutation relations of these basis are collected as follows:

$$\begin{aligned} [h, E] &= E, \quad [h, F] = -F, \quad [h, e] = 2e, \quad [h, f] = -2f, \\ [e, E] &= 0, \quad [e, F] = -E, \quad [e, f] = h, \\ [f, E] &= -F, \quad [f, F] = 0, \\ [E, E] &= 2e, \quad [E, F] = h, \quad [F, F] = -2f. \end{aligned}$$

It is not hard to see that the restricted enveloping algebra $\mathbf{u}(\mathfrak{osp}(1|2))$ of $\mathfrak{osp}(1|2)$ is generated by even element h and odd elements E, F with relations

$$\begin{aligned} EF + FE &= h, \quad hE - Eh = E, \quad hF - Fh = -F, \\ E^{2p} &= F^{2p} = 0, \quad h^p = h. \end{aligned}$$

The structures of simple modules and Verma modules have been given in a more general context in [33]. Let's recall them. For any $0 \leq \lambda \leq p-1$, we define a $2\lambda+1$ -dimensional $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -module V^λ as follows. This module has a basis $v_0, v_1, \dots, v_{2\lambda}$ and the actions of the generators are given by the following rules

$$(5.7) \quad hv_i = (\lambda - i)v_i, \quad Ev_i = \begin{cases} -\frac{i}{2}v_{i-1}, & \text{if } i \text{ is even} \\ (\lambda - \frac{i-1}{2})v_{i-1}, & \text{if } i \text{ is odd,} \end{cases} \quad Fv_i = v_{i+1}$$

where $i = 0, 1, \dots, 2\lambda$ and $v_{-1} = v_{2\lambda+1} = 0$. By Proposition 6.3 in [33], $\{V^\lambda | 0 \leq \lambda \leq p-1\}$ forms a complete non-redundant list of simple $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -modules. The graphical representation for V^1 is indicated as follows. Similar to the case of \mathfrak{sl}_2 , arrows and dotted ones show the actions of E and F respectively.

$$\begin{array}{c} \cdot \\ \uparrow \\ \cdot \\ \downarrow \\ \cdot \\ \uparrow \\ \cdot \\ \downarrow \\ \cdot \end{array}$$

Let \mathbf{u}_+ and \mathbf{u}_- be the subalgebras of $|\mathbf{u}(\mathfrak{osp}(1|2))|$ generated by h, E and h, F respectively. Also, for any $0 \leq \lambda \leq p-1$, we have the Verma modules W^λ and \tilde{W}^λ which are free over \mathbf{u}_+ and \mathbf{u}_- respectively. They are given by the following rules.

W^λ :

$$(5.8) \quad hv_i = (\lambda - i)v_i, \quad Ev_i = \begin{cases} -\frac{i}{2}v_{i-1}, & \text{if } i \text{ is even} \\ (\lambda - \frac{i-1}{2})v_{i-1}, & \text{if } i \text{ is odd,} \end{cases} \quad Fv_i = v_{i+1},$$

\tilde{W}^λ :

$$(5.9) \quad hv_i = (i - \lambda)v_i, \quad Ev_i = v_{i+1}, \quad Fv_i = \begin{cases} \frac{i}{2}v_{i-1}, & \text{if } i \text{ is even} \\ (\frac{i-1}{2} - \lambda)v_{i-1}, & \text{if } i \text{ is odd,} \end{cases}$$

where $i = 0, 1, \dots, 2p-1$ and $v_{-1} = v_{2p} = 0$. For $p = 3, \lambda = 1$, their graphical representations are as follows.

$$W^1 : \quad \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot$$

$$\tilde{W}^1 : \quad \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot \xrightarrow{\cdot} \cdot$$

Clearly, all Verma modules have dimensions $2p$ and we have the following non-split extensions:

$$0 \rightarrow V^{p-1-\lambda} \rightarrow W^\lambda \rightarrow V^\lambda \rightarrow 0,$$

$$0 \rightarrow V^{p-1-\lambda} \rightarrow \tilde{W}^\lambda \rightarrow V^\lambda \rightarrow 0,$$

for $0 \leq \lambda \leq p-1$.

Remark 5.3. Contrast to the \mathfrak{sl}_2 case, the Verma modules $W^{\frac{p-1}{2}}, \tilde{W}^{\frac{p-1}{2}}$ are special. Now, $\text{Hom}_{|\mathbf{u}(\mathfrak{osp}(1|2))|}(W^{\frac{p-1}{2}}, W^{\frac{p-1}{2}}) \cong \text{Hom}_{|\mathbf{u}(\mathfrak{osp}(1|2))|}(\tilde{W}^{\frac{p-1}{2}}, \tilde{W}^{\frac{p-1}{2}}) \cong \kappa[x]/(x^2)$ while all Verma modules of $\mathbf{u}(\mathfrak{sl}_2)$ are bricks, that is, their endomorphism rings are isomorphic to κ .

5.3.2. Projective modules. Inspired by the case of \mathfrak{sl}_2 and the work given by Xiao [36], we define the following modules, which will be shown to form a complete list of indecomposable projective $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -modules.

For any $0 \leq \lambda \leq p-1$, we define an $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -module, denoted by $P^{p-1-\lambda}$, by the following rules. As a space, it has a basis consisting of $\{b_i, a_i, x_j, y_j | 0 \leq i \leq 2p-2-2\lambda, 0 \leq j \leq 2\lambda\}$. The actions of h, E, F are given by:

$$(5.10) \quad hb_i = (p-1-\lambda-i)b_i, \quad Fb_i = b_{i+1},$$

$$(5.11) \quad Eb_i = \begin{cases} -\frac{i}{2}b_{i-1} + a_{i-1}, & \text{if } i \text{ is even} \\ (p-1-\lambda-\frac{i-1}{2})b_{i-1} - a_{i-1}, & \text{if } i \text{ is odd;} \end{cases}$$

$$(5.12) \quad hy_j = (j-\lambda)y_j, \quad Ey_j = y_{j+1},$$

$$(5.13) \quad Fy_j = \begin{cases} \frac{j}{2}y_{j-1}, & \text{if } j \text{ is even} \\ (\frac{j-1}{2}-\lambda)y_{j-1}, & \text{if } j \text{ is odd;} \end{cases}$$

$$(5.14) \quad hx_j = (\lambda-j)x_j, \quad Fx_j = x_{j+1},$$

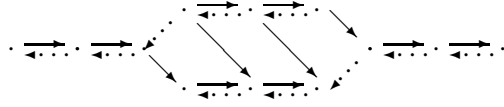
$$(5.15) \quad Ex_j = \begin{cases} -\frac{j}{2}x_{j-1}, & \text{if } j \text{ is even} \\ (\lambda-\frac{j-1}{2})x_{j-1}, & \text{if } j \text{ is odd;} \end{cases}$$

$$(5.16) \quad ha_i = (p-1-\lambda-i)a_i, \quad Fa_i = a_{i+1},$$

$$(5.17) \quad Ea_i = \begin{cases} -\frac{i}{2}a_{i-1}, & \text{if } i \text{ is even} \\ (p-1-\lambda-\frac{i-1}{2})a_{i-1}, & \text{if } i \text{ is odd,} \end{cases}$$

where $b_{2p-1-2\lambda} = y_{2\lambda}$, $y_{2\lambda+1} = a_{2p-1-2\lambda}$, $x_{2\lambda+1} = a_0$, $b_{-1} = x_{2\lambda}$.

The graphical description of the $P^{p-1-\lambda}$ (for $p=3, \lambda=1$) is indicated as follows.



Proposition 5.4. *The set $\{P^\lambda | 0 \leq \lambda \leq p-1\}$ gives a complete list of non-isomorphic indecomposable projective $|\mathfrak{u}(\mathfrak{osp}(1|2))|$ -modules. All of them have dimensions $4p$.*

Proof. The second statement is obvious. For any $0 \leq \lambda \leq p-1$, $P^{p-1-\lambda}$ is clearly indecomposable and its head is isomorphic to $V^{p-1-\lambda}$. Owing to the classification of simple $|\mathfrak{u}(\mathfrak{osp}(1|2))|$ -modules, the conclusion is proved provided that we can show $P^{p-1-\lambda}$ is projective. Actually, from $E^2 = e$ and $F^2 = -f$ in $|\mathfrak{u}(\mathfrak{osp}(1|2))|$, one can write the actions of e, f on the basis given in (5.10)-(5.17) directly:

$$\begin{aligned}
 eb_i &= \begin{cases} -\frac{i}{2}(p-\lambda-\frac{i}{2})b_{i-2} + (p-\lambda)a_{i-2}, & \text{if } i \text{ is even} \\ -\frac{i-1}{2}(p-\lambda-1-\frac{i-1}{2})b_{i-2} + (p-1-\lambda)a_{i-2}, & \text{if } i \text{ is odd,} \end{cases} \quad fb_i = -b_{i+2}; \\
 ea_i &= \begin{cases} -\frac{i}{2}(p-\lambda-\frac{i}{2})a_{i-2}, & \text{if } i \text{ is even} \\ -\frac{i-1}{2}(p-\lambda-1-\frac{i-1}{2})a_{i-2}, & \text{if } i \text{ is odd,} \end{cases} \quad fa_i = -a_{i+2}; \\
 ey_j &= y_{j+2}, \quad fy_j = \begin{cases} -\frac{j}{2}(\frac{j}{2}-\lambda-1)y_{j-2}, & \text{if } j \text{ is even} \\ -\frac{j-1}{2}(\frac{j-1}{2}-\lambda)y_{j-2}, & \text{if } j \text{ is odd,} \end{cases} \\
 ex_j &= \begin{cases} -\frac{j}{2}(\lambda+1-\frac{j}{2})x_{j-2}, & \text{if } j \text{ is even} \\ -\frac{j-1}{2}(\lambda-\frac{j-1}{2})x_{j-2}, & \text{if } j \text{ is odd,} \end{cases} \quad , fx_j = -x_{j+2}
 \end{aligned}$$

for $0 \leq i \leq 2p-2\lambda-2$ and $0 \leq j \leq 2\lambda$. Denote the restriction of $P^{p-1-\lambda}$ to $\mathfrak{u}(\mathfrak{sl}_2)$ by $P^{p-1-\lambda}|_{\mathfrak{u}(\mathfrak{sl}_2)}$. Then, it is not hard to see that

$$P^{p-1-\lambda}|_{\mathfrak{u}(\mathfrak{sl}_2)} \cong P_0^{p-1-\lambda} \oplus P_0^{p-2-\lambda}$$

if $\lambda \neq 0, p-1$, and

$$P^0|_{\mathfrak{u}(\mathfrak{sl}_2)} \cong P_0^0 \oplus 2V_0^{p-1}, \quad P^{p-1}|_{\mathfrak{u}(\mathfrak{sl}_2)} \cong P_0^{p-2} \oplus 2V_0^{p-1}.$$

As a conclusion, the restriction $P^{p-1-\lambda}|_{\mathfrak{u}(\mathfrak{sl}_2)}$ is projective for all $0 \leq \lambda \leq p-1$. Therefore, Lemma 5.2 implies $P^{p-1-\lambda}$ itself is projective. \square

An indecomposable projective module corresponds to an extension of Verma modules. Indeed, for any $0 \leq \lambda \leq p-1$ we have the following non-split exact sequences

$$0 \rightarrow W^\lambda \rightarrow P^{p-1-\lambda} \rightarrow W^{p-1-\lambda} \rightarrow 0,$$

$$0 \rightarrow \tilde{W}^\lambda \rightarrow P^{p-1-\lambda} \rightarrow \tilde{W}^{p-1-\lambda} \rightarrow 0.$$

This verifies and strengthens the Proposition 6.3 (iii) in [33], which states P^λ has a Verma filtration with W^λ and $W^{p-1-\lambda}$ as subquotients.

5.3.3. *Blocks and Auslander-Reiten quivers.* By above proposition and the structures of projective modules, we know that only V^λ and $V^{p-1-\lambda}$ are composition factors of P^λ for $0 \leq \lambda \leq p-1$. Thus there are exactly $\frac{p+1}{2}$ blocks and $V^\lambda, V^{p-1-\lambda}$ belong to the same block for $0 \leq \lambda \leq \frac{p-1}{2}$. In particular, the block containing $V^{\frac{p-1}{2}}$ is primary, that is, it has only one simple module. Our next aim is to describe the basic algebras of these blocks using quivers with relations. For more on quivers and related terminologies, see [2].

Take an $\lambda \in \{0, 1, \dots, p-1\}$. By the standard methods using in representation theory of finite dimensional algebras [2], the basic algebra of the block containing V^λ is isomorphic to

$$\text{End}_{|\mathbf{u}(\mathfrak{osp}(1|2))|}(P^\lambda \oplus P^{p-1-\lambda})$$

if $\lambda \neq \frac{p-1}{2}$ and isomorphic to

$$\text{End}_{|\mathbf{u}(\mathfrak{osp}(1|2))|}(P^{\frac{p-1}{2}})$$

otherwise.

Define Λ_2 to be the algebra given by the following quiver and relations

$$\begin{array}{c} \xrightarrow{y_1} \\ \cdot \xrightarrow{x_1} \cdot \\ \xleftarrow{x_2} \\ \xleftarrow{y_2} \end{array} \quad \begin{array}{l} x_i x_j = y_i y_j \\ x_i y_j = y_i x_j = 0 \end{array} \quad \text{for } 1 \leq i \neq j \leq 2,$$

and Λ_1 given by

$$\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \quad x^2 = y^2, \quad xy = yx = 0.$$

Lemma 5.5. *Assume that $0 \leq \lambda \leq p-1$.*

- (1) *If $\lambda \neq \frac{p-1}{2}$, then $\text{End}_{|\mathbf{u}(\mathfrak{osp}(1|2))|}(P^\lambda \oplus P^{p-1-\lambda}) \cong \Lambda_2$.*
- (2) $\text{End}_{|\mathbf{u}(\mathfrak{osp}(1|2))|}(P^{\frac{p-1}{2}}) \cong \Lambda_1$.

Proof. We only prove (1) since (2) can be proved similarly. For (1), we can represent projective modules $P^{p-1-\lambda}$ and P^λ by using the following graphs:

$$\begin{array}{cc} P^{p-1-\lambda} : & \begin{array}{c} V^{p-1-\lambda} \bullet \\ \swarrow \quad \searrow \\ V^{\lambda \circ} \quad \circ V^\lambda \\ \swarrow \quad \searrow \\ V^{p-1-\lambda} \bullet \end{array} & P^\lambda : & \begin{array}{c} V^\lambda_\circ \\ \swarrow \quad \searrow \\ V^{p-1-\lambda} \bullet \quad \bullet V^{p-1-\lambda} \\ \swarrow \quad \searrow \\ V^\lambda_\circ \end{array} \end{array}$$

From this, one can see that there are exactly two non-trivial linear independent $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -morphisms from $P^{p-1-\lambda}$ to P^λ :

$$x_1 : P^{p-1-\lambda} \rightarrow \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} \quad y_1 : P^{p-1-\lambda} \rightarrow \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}$$

Similarly, we also have two non-trivial linear independent $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -morphisms x_2, y_2 from P^λ to $P^{p-1-\lambda}$:

$$x_2 : P^\lambda \rightarrow \begin{array}{c} \circ \\ \diagup \\ \bullet \end{array} \quad y_2 : P^\lambda \rightarrow \begin{array}{c} \circ \\ \diagdown \\ \bullet \end{array}$$

Clearly, such maps indeed generate $\text{End}_{|\mathbf{u}(\mathfrak{osp}(1|2))|}(P^\lambda \oplus P^{p-1-\lambda})$ and exactly satisfy the relations in the definition of Λ_2 . \square

Summarizing, we have proved the following.

Proposition 5.6. *Let κ be an algebraically closed field of characteristic $p > 2$, and $\mathbf{u}(\mathfrak{osp}(1|2))$ the restricted enveloping algebra of Lie superalgebra $\mathfrak{osp}(1|2)$ over κ . Then*

- (1) *The algebra $|\mathbf{u}(\mathfrak{osp}(1|2))|$ has p isomorphism classes of simple modules, i.e. V^λ for $0 \leq \lambda \leq p-1$.*
- (2) *The algebra $|\mathbf{u}(\mathfrak{osp}(1|2))|$ has $\frac{p+1}{2}$ blocks.*
- (3) *The block containing $V^{\frac{p-1}{2}}$ is primary and its basic algebra is isomorphic to Λ_1 .*
- (4) *For any $0 \leq \lambda < \frac{p-1}{2}$, the simple modules V^λ and $V^{p-1-\lambda}$ belong to the same block, denoted by $B(\lambda)$, whose basic algebra is isomorphic to Λ_2 .*

Remark 5.7. (1) Let A be an artin algebra. Operating it by its dual $D(A) := \text{Hom}_\kappa(A, \kappa)$, one can get a new algebra $T(A)$, called the *trivial extension* of A . By definition, the underlying vector space of $T(A) = A \oplus D(A)$ and the multiplication is given by

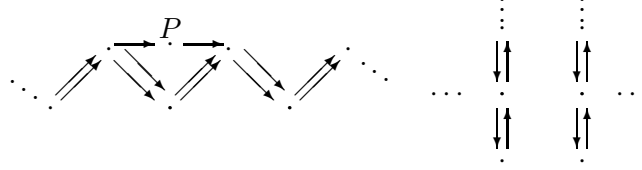
$$(a, d)(a', d') = (aa', da' + ad')$$

for $a, a' \in A$, $d, d' \in D(A)$ by noting $D(A)$ is an A - A -bimodule in an obvious way. It is not hard to see that Λ_2 is indeed the trivial extension of the Kronecker algebra, that is, the path algebra of the quiver

$$\begin{array}{ccc} & \xrightarrow{\quad} & \\ & \xrightarrow{\quad} & \end{array}$$

(2) By the Example in Section 4 in [14], $|\mathbf{u}(\mathfrak{osp}(1|2))|$ is a tame algebra, which is a direct consequence of our results now. Moreover, one can see that all blocks of $|\mathbf{u}(\mathfrak{osp}(1|2))|$ are tame. This is not the case for $\mathbf{u}(\mathfrak{sl}_2)$, which has exactly one block of finite representation type.

The categories of finite dimensional representations over algebras Λ_1 and Λ_2 had been well understood. Recall a graph is called a *tube* if it is isomorphic to $\mathbb{Z}A_\infty/n$ for some positive integer n and n is called the *rank* of this tube. A rank 1 tube is said to be *homogeneous*. For details about Auslander-Reiten quivers and translation quivers, see Chapter VII in [2] and [28]. The Auslander-Reiten quiver of Λ_1 can be drawn as follows.



A $\mathbb{P}^1\kappa$ family of homogeneous tubes

The Auslander-Reiten quiver of Λ_2 is just the double of that of Λ_1 .

5.4. Finite dimensional indecomposable modules. Inspired by the forms of the Auslander-Reiten quivers of the basic algebras of its blocks and the familiar representation theory of Λ_2 and Λ_1 , we can construct all the indecomposable representations of $|\mathbf{u}(\mathfrak{osp}(1|2))|$ now.

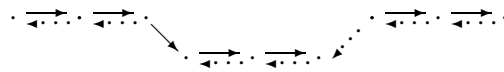
5.4.1. $V^\lambda(n)$ and $\tilde{V}^\lambda(n)$. For any positive integer n and $0 \leq \lambda \leq p-1$, the basis of $V^\lambda(n)$ is

$$\{a_u(m-1), e_v(m) | 0 \leq m \leq n, 0 \leq u \leq 2p-2\lambda-2, 0 \leq v \leq 2\lambda\}$$

with actions given by

$$\begin{aligned} he_v(m) &= (\lambda - v)e_v(m), \quad Fe_v(m) = e_{v+1}(m), \\ Ee_v(m) &= \begin{cases} -\frac{v}{2}e_{v-1}(m) + \delta_{v0}a_{2p-2\lambda-2}(m), & \text{if } v \text{ is even} \\ (\lambda - \frac{v-1}{2})e_{v-1}(m), & \text{if } v \text{ is odd,} \end{cases} \\ ha_u(m-1) &= (p-1-\lambda-u)a_u(m-1), \quad Fa_u(m-1) = a_{u+1}(m-1), \\ Ea_u(m-1) &= \begin{cases} -\frac{u}{2}a_{u-1}(m-1), & \text{if } u \text{ is even} \\ (p-1-\lambda-\frac{u-1}{2})a_{u-1}(m-1), & \text{if } u \text{ is odd,} \end{cases} \end{aligned}$$

where $a_u(-1) = a_u(n) = 0$ for $0 \leq u \leq 2p-2\lambda-2$, $a_{-1}(m-1) = a_{2p-2\lambda-1}(m-1) = 0$ for $1 \leq m \leq n$ and $e_{2\lambda+1}(m) = a_0(m-1)$ for $1 \leq m \leq n$. The following is the graphical description of $V^\lambda(n)$ in the case $n=1, \lambda=1$ and $p=3$:



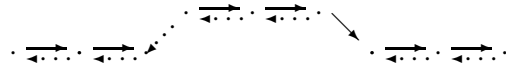
For any positive integer n and $0 \leq \lambda \leq p-1$, the basis of $\tilde{V}^\lambda(n)$ is

$$\{a_u(m-1), e_v(m) | 0 \leq m \leq n, 0 \leq u \leq 2p-2\lambda-2, 0 \leq v \leq 2\lambda\}$$

with actions given by

$$\begin{aligned}
he_v(m) &= (\lambda - v)e_v(m), \quad Fe_v(m) = e_{v+1}(m), \\
Ee_v(m) &= \begin{cases} -\frac{v}{2}e_{v-1}(m), & \text{if } v \text{ is even} \\ (\lambda - \frac{v-1}{2})e_{v-1}(m), & \text{if } v \text{ is odd,} \end{cases} \\
ha_u(m-1) &= (p-1-\lambda-u)a_u(m-1), \quad Fa_u(m-1) = a_{u+1}(m-1), \\
Ea_u(m-1) &= \begin{cases} -\frac{u}{2}a_{u-1}(m-1) + \delta_{u0}e_{2\lambda}(m), & \text{if } u \text{ is even} \\ (p-1-\lambda-\frac{u-1}{2})a_{u-1}(m-1), & \text{if } u \text{ is odd,} \end{cases}
\end{aligned}$$

where $a_u(-1) = a_u(n) = 0$ for $0 \leq u \leq 2p-2\lambda-2$, $e_{-1}(m-1) = e_{2\lambda+1}(m-1) = 0$ for $0 \leq m \leq n$ and $e_0(m-1) = a_{2p-2\lambda-1}(m-1)$ for $1 \leq m \leq n$. The following is the graphical description of $\tilde{V}^\lambda(n)$ in the case $n = 1, \lambda = 1$ and $p = 3$:



For $n \geq 1$, the induced Auslander-Reiten sequences are

$$0 \rightarrow V^\lambda(n) \rightarrow V^\lambda(n+1) \oplus V^\lambda(n+1) \rightarrow V^\lambda(n+2) \rightarrow 0,$$

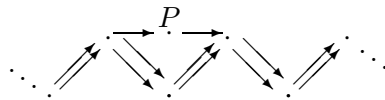
$$0 \rightarrow \tilde{V}^\lambda(n+2) \rightarrow \tilde{V}^\lambda(n+1) \oplus \tilde{V}^\lambda(n+1) \rightarrow \tilde{V}^\lambda(n) \rightarrow 0,$$

$$0 \rightarrow \tilde{V}^\lambda(1) \rightarrow V^\lambda \oplus P^{p-1-\lambda} \oplus V^\lambda \rightarrow V^\lambda(1) \rightarrow 0.$$

Note that $V^\lambda(0) = V^\lambda = \tilde{V}^\lambda(0)$. The Auslander-Reiten translation is given by

$$\tau V^\lambda(n+2) = V^\lambda(n), \quad \tau \tilde{V}^\lambda(n) = \tilde{V}^\lambda(n+2), \quad \text{for } n \geq 0 \text{ and } \tau V^\lambda(1) = \tilde{V}^\lambda(1).$$

It is not hard to see that they indeed give the preprojective component, showing as follows, of the Auslander-Reiten quiver described after Remark 5.7.



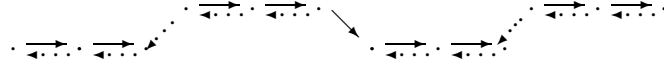
5.4.2. $W^\lambda(n)$ and $\tilde{W}^\lambda(n)$. For any positive integer n and $0 \leq \lambda \leq p-1$, the basis of $W^\lambda(n)$ is

$$\{e_u(m) | 1 \leq m \leq n, 0 \leq u \leq 2p-1\}$$

with actions given by

$$\begin{aligned} he_u(m) &= (\lambda - u)e_u(m), \quad Fe_u(m) = e_{u+1}(m), \\ Ee_u(m) &= \begin{cases} -\frac{u}{2}e_{u-1}(m) + \delta_{u0}e_{2p-1}(m+1), & \text{if } u \text{ is even} \\ (\lambda - \frac{u-1}{2})e_{u-1}(m), & \text{if } u \text{ is odd,} \end{cases} \end{aligned}$$

where $e_u(n+1) = 0$ for $0 \leq u \leq 2p-1$, $e_{2p}(m) = 0$ for $1 \leq m \leq n$. The following is the graphical description of $W^\lambda(n)$ in the case $n=2, \lambda=1$ and $p=3$:



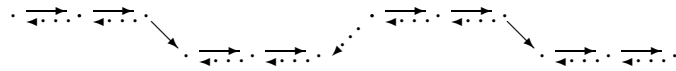
For any positive integer n and $0 \leq \lambda \leq p-1$, the basis of $\tilde{W}^\lambda(n)$ is

$$\{f_u(m) | 1 \leq m \leq n, 0 \leq u \leq 2p-1\}$$

with actions given by

$$\begin{aligned} hf_u(m) &= (\lambda - u)f_u(m), \quad Ef_u(m) = f_{u+1}(m), \\ Ff_u(m) &= \begin{cases} \frac{u}{2}f_{u-1}(m) + \delta_{u0}f_{2p-1}(m-1), & \text{if } u \text{ is even} \\ (\frac{u-1}{2} - \lambda)f_{u-1}(m), & \text{if } u \text{ is odd,} \end{cases} \end{aligned}$$

where $f_u(0) = 0$ for $0 \leq u \leq 2p-1$, $f_{2p}(m) = 0$ for $1 \leq m \leq n$. The following is the graphical description of $\tilde{W}^\lambda(n)$ in the case $n=2, \lambda=1$ and $p=3$:



For $n \geq 1$, the induced Auslander-Reiten sequences are

$$0 \rightarrow W^\lambda(n) \rightarrow W^\lambda(n+1) \oplus W^\lambda(n-1) \rightarrow W^\lambda(n) \rightarrow 0,$$

$$0 \rightarrow \tilde{W}^\lambda(n) \rightarrow \tilde{W}^\lambda(n+1) \oplus \tilde{W}^\lambda(n-1) \rightarrow \tilde{W}^\lambda(n) \rightarrow 0.$$

Here we define $W^\lambda(0) = \tilde{W}^\lambda(0) = 0$. The Auslander-Reiten translation is given by

$$\tau W^\lambda(n) = W^\lambda(n), \quad \tau \tilde{W}^\lambda(n) = \tilde{W}^\lambda(n) \text{ for } n \geq 1.$$

$$\{e_u(m), \hat{e}_u(m) | 1 \leq m \leq n, 0 \leq u \leq 2p-1\}$$
$$\begin{aligned} he_u(m) &= (\lambda - u)e_u(m), \quad Fe_u(m) = e_{u+1}(m), \\ Ee_u(m) &= \begin{cases} -\frac{u}{2}e_{u-1}(m) + s_1\delta_{u0}\hat{e}_{2p-1}(m) + \delta_{u0}\hat{e}_{2p-1}(m-1), & \text{if } u \text{ is even} \\ (\lambda - \frac{u-1}{2})e_{u-1}(m), & \text{if } u \text{ is odd,} \end{cases} \\ h\hat{e}_u(m) &= (\lambda - u)\hat{e}_u(m), \quad F\hat{e}_u(m) = \hat{e}_{u+1}(m), \\ E\hat{e}_u(m) &= \begin{cases} -\frac{u}{2}\hat{e}_{u-1}(m) + s_2\delta_{u0}e_{2p-1}(m) + \delta_{u0}e_{2p-1}(m-1), & \text{if } u \text{ is even} \\ (\lambda - \frac{u-1}{2})\hat{e}_{u-1}(m), & \text{if } u \text{ is odd,} \end{cases} \end{aligned}$$
$$0 \rightarrow T^\lambda(s, n) \rightarrow T^\lambda(s, n+1) \oplus T^\lambda(s, n-1) \rightarrow T^\lambda(s, n) \rightarrow 0,$$
$$\tau T^\lambda(s, n) = T^\lambda(s, n) \quad \text{for } n \geq 1.$$
$$\varphi(e_u) = c_1 e'_u, \quad \varphi(\hat{e}_u(1)) = c_2 \hat{e}'_u(1),$$

or

$$\varphi(e_u(1)) = c_1 \hat{e}'_u(1), \quad \varphi(\hat{e}_u(1)) = c_2 e'_u(1),$$

for some $c_1, c_2 \in \kappa^*$. In the first case, by $\varphi(Ee_0(1)) = E\varphi(e_0(1))$ and $\varphi(E\hat{e}_0(1)) = E\varphi(\hat{e}_0(1))$, we have

$$s_1 c_2 = c_1 t_1, \quad s_2 c_1 = c_2 t_2$$

which implies $\frac{s_1}{t_1} = \frac{c_1}{c_2} = \frac{t_2}{s_2}$. In the second case, also from $\varphi(Ee_0(1)) = E\varphi(e_0(1))$ and $\varphi(E\hat{e}_0(1)) = E\varphi(\hat{e}_0(1))$, one can show that

$$s_1 c_2 = c_1 t_2, \quad s_2 c_1 = c_2 t_1$$

and so $\frac{s_1}{t_2} = \frac{c_1}{c_2} = \frac{t_1}{s_2}$.

“ \Leftarrow ” Conversely, define

$$\varphi : T^\lambda(s, 1) \rightarrow T^\lambda(t, 1), \quad e_u(1) \mapsto \frac{s_1}{t_1} e'_u(1), \quad \hat{e}_u(1) \mapsto \hat{e}'_u(1)$$

for $0 \leq u \leq 2p-1$. It is direct to show that φ is a morphism and bijective. Thus $T^\lambda(s, 1) \cong T^\lambda(t, 1)$. From the Auslander-Reiten sequences we constructed, $T^\lambda(s, n) \cong T^\lambda(t, n)$ for any $n \geq 1$.

□

Not that $\frac{s_1}{t_1} = \frac{t_2}{s_2}$ is equivalent to $s_1 s_2 = t_1 t_2$. For any $c \in \kappa^*$, define $T_c^\lambda(n)$ to be any one of $T^\lambda(s, n)$ satisfying $s_1 s_2 = c$. Proposition 5.8 implies that $\{T_c^\lambda(n) | c \in \kappa^*, n \geq 1\}$ forms a complete set of representatives of modules $\{T^\lambda(s, n) | s = (s_1, s_2) \in \kappa^* \times \kappa^*, n \geq 1\}$ for any fixed $\lambda \in \{0, 1, \dots, 2p-1\}$.

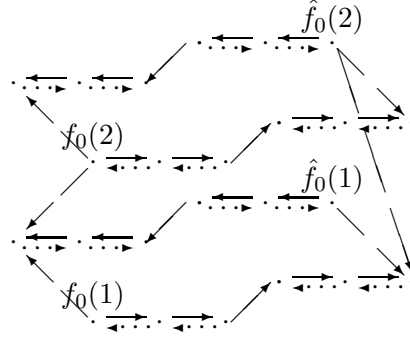
Remark 5.9. Similar to the case of 5.4.1, 5.4.2, one also can define an indecomposable module $\tilde{T}^\lambda(s, n)$ for $n \in \mathbb{Z}^+$, $0 \leq \lambda \leq p-1$ and $s = (s_1, s_2) \in \kappa^* \times \kappa^*$. Similarly, the basis of $\tilde{T}^\lambda(s, n)$ is

$$\{f_u(m), \hat{f}_u(m) | 1 \leq m \leq n, 0 \leq u \leq 2p-1\}$$

with actions given by

$$\begin{aligned} hf_u(m) &= (u - \lambda)e_u(m), \quad Ef_u(m) = f_{u+1}(m), \\ Ff_u(m) &= \begin{cases} \frac{u}{2}f_{u-1}(m) + s_1\delta_{u0}\hat{f}_{2p-1}(m) + \delta_{u0}\hat{f}_{2p-1}(m-1), & \text{if } u \text{ is even} \\ (\frac{u-1}{2} - \lambda)f_{u-1}(m), & \text{if } u \text{ is odd,} \end{cases} \\ h\hat{f}_u(m) &= (u - \lambda)\hat{f}_u(m), \quad E\hat{f}_u(m) = \hat{f}_{u+1}(m), \\ F\hat{f}_u(m) &= \begin{cases} \frac{u}{2}\hat{f}_{u-1}(m) + s_2\delta_{u0}f_{2p-1}(m) + \delta_{u0}f_{2p-1}(m-1), & \text{if } u \text{ is even} \\ (\frac{u-1}{2} - \lambda)\hat{f}_{u-1}(m), & \text{if } u \text{ is odd,} \end{cases} \end{aligned}$$

where $f_u(0) = \hat{f}_u(0) = 0$ for $0 \leq u \leq 2p-1$, and $f_{2p}(m) = \hat{f}_{2p}(m) = 0$ for $1 \leq m \leq n$. The following is the graphical description of $\tilde{T}^\lambda(s, n)$ in the case $n = 2, \lambda = 1$ and $p = 3$:



But they will not provide new modules, which is a result of our next theorem.

Combing the Auslander-Reite sequences constructed for $W^\lambda(n)$, $\tilde{W}^\lambda(n)$, $T^\lambda(s, n)$ and Proposition 5.8, we have $\mathbb{P}^1\kappa$ family of homogeneous tubes now for any fixed $\lambda \in \{0, 1, \dots, 2p-1\}$. Comparing with the Auslander-Reiten quiver constructed after Remark 5.7, we can summarize our works up to now into the following result.

Theorem 5.10. *The modules*

- (1) P^λ for $0 \leq \lambda \leq p-1$,
- (2) $V^\lambda(n), \tilde{V}^\lambda(n)$ for $0 \leq \lambda \leq p-1, n \geq 0$,
- (3) $W^\lambda(n), \tilde{W}^\lambda(n)$ for $0 \leq \lambda \leq p-1, n \geq 1$, and
- (4) $T_c^\lambda(n)$ for $c \in \kappa^*, 0 \leq \lambda \leq p-1, n \geq 1$,

form a complete list of all finite dimensional indecomposable modules of $|\mathbf{u}(\mathfrak{osp}(1|2))|$ up to isomorphism.

It is not hard to see that all indecomposable $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -modules we constructed are indeed supermodules naturally. For example, for the simple module V^λ defined as in (5.7), we can set $v_0, v_2, \dots, v_{2\lambda}$ to be even elements while $v_1, v_3, \dots, v_{2\lambda-1}$ be odd elements. From this, V^λ is an $\mathbf{u}(\mathfrak{osp}(1|2))$ -supermodule. Similar for other modules.

So now we can assume all modules in Theorem 5.10 are $\mathbf{u}(\mathfrak{osp}(1|2))$ -supermodules. Let Π be the *parity change functor*, by definition it just interchanges the \mathbb{Z}_2 -grading of a supermodule.

Corollary 5.11. *The modules*

- (1) $P^\lambda, \Pi(P^\lambda)$ for $0 \leq \lambda \leq p-1$,
- (2) $V^\lambda(n), \tilde{V}^\lambda(n), \Pi(V^\lambda(n)), \Pi(\tilde{V}^\lambda(n))$ for $0 \leq \lambda \leq p-1, n \geq 0$,
- (3) $W^\lambda(n), \tilde{W}^\lambda(n), \Pi(W^\lambda(n)), \Pi(\tilde{W}^\lambda(n))$ for $0 \leq \lambda \leq p-1, n \geq 1$, and
- (4) $T_c^\lambda(n), \Pi(T_c^\lambda(n))$ for $c \in \kappa^*, 0 \leq \lambda \leq p-1, n \geq 1$,

form a complete list of all finite dimensional indecomposable supermodules of $\mathbf{u}(\mathfrak{osp}(1|2))$ up to isomorphism.

Proof. It is enough to show every indecomposable supermodule M is indeed indecomposable as an $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -module. Assume now $M = M_1 \oplus M_2$ in the $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -module category. Let $\pi : \bigoplus_{i \in I} P_i \rightarrow M$ be the projective cover of M in the category of supermodules. Here we assume every P_i is indecomposable as a supermodule. By Proposition 12.2.12 in [19] and our description of projective $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -modules, all P_i are indeed indecomposable projective $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -modules. So $\bigoplus_{i \in I} P_i \rightarrow M$ is also a surjection as $|\mathbf{u}(\mathfrak{osp}(1|2))|$ -modules. Therefore, we can assume that there is subset $J \subset I$ such that $\pi(\bigoplus_{i \in J} P_i) = M_1$ and so M_1 is a supermodule. Similarly, M_2 is a supermodule too. Thus $M = M_1$ or $M = M_2$. \square

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